DOI 10.15507/2079-6900.25.202304.299-312
Original article

ISSN 2079-6900 (Print)
ISSN 2587-7496 (Online)

MSC2020 34D20

On the stability of a nonlinear nonautonomous scalar equation with variable delay

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Abstract. The stability problem of a scalar functional differential equation is a classical one. It has been most fully studied for linear equations. Modern research on modeling biological, infectious and other processes leads to the need to determine the qualitative properties of the solutions for more general equations. In this paper we study the stability and the global limit behavior of solutions to a nonlinear one-dimensional (scalar) equation with variable delay with unbounded and bounded right-hand sides. In particular, our research is reduced to a problem on the stability of a non-stationary solution of a nonlinear scalar Lotka-Volterra-type equation, on the stabilization and control of a non-stationary process described by such an equation. The problem posed is considered depending on the delay behavior: is it a bounded differentiable function or a continuous and bounded one. The study is based on the application of the Lyapunov-Krasovsky functionals method as well as the corresponding theorems on the stability of non-autonomous functional differential equations of retarded type with finite delay. Sufficient conditions are derived for uniform asymptotic stability of the zero solution, including global stability, for every continuous initial function. Using the theorem proven by one of the co-authors on the limiting behavior of solutions to a non-autonomous functional differential equation based on the Lyapunov functional with a semidefinite derivative, the properties of the solutions' attraction to the set of equilibrium states of the equation under study are obtained. In addition, illustrative examples are provided.

Keywords: nonlinear scalar differential equation, variable delay, stability, attraction of solutions, Lyapunov functional

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Оригинальная статья

УДК 517.929.4

Об устойчивости нелинейного неавтономного скалярного уравнения с переменным запаздыванием

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Аннотация. Задача устойчивости скалярного функционально-дифференциального уравнения имеет классический характер. Наиболее полно она изучена для уравнений линейного типа. Современные исследования по моделированию биологических, инфекционных и других процессов приводят к необходимости определения качественных свойств решений более общих уравнений. В данной работе изучается задача об устойчивости и глобальном предельном поведении решений нелинейного одномерного (скалярного) уравнения с переменным запаздыванием, с неограниченной и ограниченной правой частью. К такой задаче, в частности, сводятся исследования: об устойчивости нестационарного решения нелинейного скалярного уравнения типа Лотки-Вольтерра, о стабилизации и управлении нестационарным процессом, описываемым таким уравнением. Поставленная задача рассмотрена в зависимости от случаев: запаздывание является ограниченной дифференцируемой функцией или непрерывным и ограниченным. Исследование основано на применении метода функционалов Ляпунова-Красовского и соответствующих теорем об устойчивости неавтономных функционально-дифференциальных уравнений запаздывающего типа с конечным запаздыванием. Выведены достаточные условия равномерной асимптотической устойчивости нулевого решения, в том числе, глобальной при любых начальных непрерывных функциях. По теореме одного из соавторов об исследовании предельного поведения решений неавтономного функционально-дифференциального уравнения на основе функционала Ляпунова со знакопостоянной производной выводятся свойства притяжения решений к множеству состояний равновесия исследуемого уравнения. Приведены иллюстративные примеры.

Ключевые слова: нелинейное скалярное дифференциальное уравнение, переменное запаздывание, устойчивость, притяжение решений, функционал Ляпунова

Для цитирования: Хусанов Д. Х., Каххаров А. Э. Об устойчивости нелинейного неавтономного скалярного уравнения с переменным запаздыванием // Журнал Средневолжского математического общества. 2023. Т. 25, № 4. С. 299-312. DOI: https://doi.org/10.15507/2079-6900.25.202304.299-312

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1. Introduction

The needs of science and technology in the middle of the last century stimulated the intensive development of the stability theory for functional differential equations [1–4].

In the monograph [1], fundamental results on the study of scalar linear equations were obtained. The stability problem of such an equation has been the subject of active study to this day [5–10]. Equations with distributed and several discrete delays including those with unbounded one are considered. Research in [5–8], [10] was carried out by direct analyzing of the solutions including their comparison and construction of successive approximations. The same approach was used in the works [10], [11] and others to obtain the stability conditions for the zero solution of a scalar nonlinear equation. It was assumed that the right-hand side of the equation is estimated modulo by a linear function with a time-independent proportionality coefficient. In [12–14] these results are generalized in the direction of sufficient conditions for uniform asymptotic stability with a coefficient unbounded in time. For this purpose, these works developed the method of Lyapunov-Krasovsky functionals in studying the stability of non-autonomous functional differential equations with an unbounded right-hand side in time.

Modern modeling in biology and epidemiology, in the neoclassical theory of population, living systems and other systems and processes leads to the need to study the qualitative properties of solutions to nonlinear nonautonomous differential equations that have two or more stationary states, periodic or chaotic changes in variables. The application of the obtained results to such problems is indicated in the works [8], [10], [15–18]. In the works [8], [10] the stability of models of population dynamics described by the Hutchinson and Mackey-Glass equations was studied. The work [15] considers the model of a living system based on an integral equation that reduces to a non-autonomous non-linear scalar equation with a finite or distributed delay. Stationary solutions are found and the asymptotic behavior of the perturbed solutions is studied.

The presented analysis shows that the problem of nonlocal limit properties of an essentially nonlinear nonautonomous equation with delay, which has a non-unique stationary solution, remains insufficiently studied.

In this paper, we study the stability problem and the global limit behavior for the solutions of a nonlinear equation with variable delay, with unbounded and bounded right-hand sides. Sufficient conditions for uniform asymptotic stability are derived, including global, zero solution, and attraction of solutions to a set of equilibrium states. In addition, we provide illustrative examples.

2. Formulation of the stability problem

Let $\mathbb{R}$ be the real axis, and let $h_0 > 0$ be a given number. Denote by $\mathbb{C}$ the Banach space of the continuous functions $\varphi : [-h_0, 0] \to \mathbb{R}$ with the norm $\|\varphi\| = \max_{-h_0 \le s \le 0}(|\varphi(s)|)$. For a continuous function $x : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}^+ = [0, +\infty)$ define the function $x_t \in \mathbb{C}$ by the equality $x_t(s) = x(t+s)$ $(s \in [-h_0, 0])$. Denote by $\dot{x}(t)$ the right-hand derivative of the function $x = x(t)$.

Consider the functional-differential equation

$$\dot{x}(t) = a(t, x_t)g(x(t)) + b(t, x_t)q(t, x(t - h(t))), \tag{2.1}$$

where $t \in \mathbb{R}^+$; $a, b \in C(\mathbb{R}^+ \times \mathbb{C} \to \mathbb{R})$; $h \in C(\mathbb{R}^+ \to [0, h_0])$; the functions $g \in C(\mathbb{R}^+ \to \mathbb{R})$

and $q \in C(\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R})$ are such that

$$g(0) = 0,\ q(t, 0) \equiv 0.$$

Without loss of generality, assume that for each initial point $(\alpha, \varphi) \in \mathbb{R}^+ \times \mathbb{C}$ there exists a unique solution of equation (2.1) $x = x(t, \alpha, \varphi)$ that satisfies the initial condition $x_\alpha = x_\alpha(\alpha, \varphi) = \varphi$ and is defined at $t \in [\alpha - h_0, \beta)$ $(\beta > \alpha)$. Equation (2.1) has a zero solution $x(t, \alpha, 0) \equiv 0$ $(\forall t \geq \alpha - h_0)$.

Consider the stability problem of the zero solution $x \equiv 0$ of equation (2.1) and the limit behavior of its bounded solutions $x = x(t, \alpha, \varphi)$, $|x(t, \alpha, \varphi)| \leq H = const$ $(t \geq \alpha - h_0)$.

For convenience sake, let us introduce the class $\mathcal{K}$ of Hahn-type functions [4] which are continuous and strictly monotonically increasing functions $d_i : \mathbb{R}^+ \to \mathbb{R}^+$, $d_i(0) = 0$, $i = 1, 2, \ldots$.

3. On the solution to the stability problem without the boundedness assumption of the right-hand side of equation (2.1)

Let us study the above formulated stability problem under the following assumptions about the functions $g = g(x)$ and $h = h(t)$

$$|q(t, x)| \leq \mu(t, H_1)|g(x)|,\ |g(x)| \leq L(H_1)|x| \quad \forall (t, x) \in \mathbb{R}^+ \times \{|x| \leq H_1 = const > 0\}, \tag{3.1}$$

$$h \in C^1(\mathbb{R}^+ \to [0, h_0]), \quad \dot{h}(t) = \frac{dh(t)}{dt} \leq 1 - h_1, \quad h_1 = const > 0. \tag{3.2}$$

Theorem 3.1. *Assume that the following conditions hold:*

1) $g(x)x > 0, \quad \forall x \in \{0 < |x| \leq H_0 < H_1\}$;

2) $|b(t, \varphi)|\mu(t, H_1) \leq a_0 l(\varphi)(1 - \dot{h}(t))^{1/2}$, $l(\varphi) \geq 0$, $A(t, \varphi) = 2a(t, \varphi) + a_0 l^2(\varphi) \leq -a_0 - \varepsilon_0$ $(a_0 > 0,\ \varepsilon_0 > 0)$ $\forall (t, \varphi) \in \mathbb{R}^+ \times \{\varphi \in \mathbb{C} : \|\varphi\| \leq H_1\}$.

Then the solution $x = 0$ of equation (2.1) is uniformly asymptotically stable.

Proof. Let us introduce the following function and functional

$$G(x) = \int_0^x g(\tau)d\tau, \quad V(t, \varphi) = G(\varphi(0)) + \frac{1}{2}a_0 \int_{-h(t)}^{0} g^2(\varphi(\tau))d\tau. \tag{3.3}$$

It follows from condition 1 of the theorem that there exists a function $d_1 \in \mathcal{K}$ such that

$$V(t, \varphi) \geq d_1(|\varphi(0)|).$$

Using inequalities from (3.1) we find the estimate

$$V(t, \varphi) \leq G(\varphi(0)) + \frac{1}{2}a_0 \int_{-h(t)}^{0} g^2(\varphi(\tau))d\tau \leq \frac{1}{2}L(H_1)|\varphi(0)|^2 + \frac{1}{2}a_0 L^2(H_1) \int_{-h_0}^{0} \varphi^2(\tau)d\tau \leq$$

$$\leq \frac{1}{2}[L(H_1)|\varphi(0)|^2 + a_0 L^2(H_1)|||\varphi|||^2], \tag{3.4}$$

where the norm

$$|||\varphi||| = \left(\int\limits_{-h_0}^{0} \varphi^2(\tau) d\tau \right)^{1/2}$$

is introduced in accordance with the theorem from [12].

According to inequalities (3.1), (3.2), and using the second inequality from condition 2 of the theorem, one can obtain the following estimation for the time derivative of the functional V due to the equation (2.1)

$$\dot{V}(t,\varphi) = (a(t,\varphi)+\frac{1}{2}a_0)g^2(\varphi(0))+b(t,\varphi)g(\varphi(0))q(t,\varphi(-h(t)))-\frac{1}{2}a_0(1-\dot{h}(t))g^2(\varphi(-h(t))) \leq$$

$$\leq (a(t,\varphi)+\frac{1}{2}a_0)g^2(\varphi(0)) + |b(t,\varphi)||g(\varphi(0))||q(t,\varphi(-h(t)))| - \frac{a_0}{2}(1-\dot{h}(t))g^2(\varphi(-h(t))) \leq$$

$$\leq (a(t,\varphi)+\frac{1}{2}a_0)g^2(\varphi(0)) + a_0 l(\varphi)(1-\dot{h}(t))^{1/2}|g(\varphi(0))|g(\varphi(-h(t)))-$$

$$-\frac{a_0}{2}(1-\dot{h}(t))g^2(\varphi(-h(t))) \leq \frac{1}{2}(A(t,\varphi)+a_0)g^2(\varphi(0)) \leq -\frac{1}{2}\varepsilon_0 g^2(\varphi(0)).$$

Thus, all the assumptions of the corresponding theorem from [12] hold. Hence, the theorem is proved.

The following result holds.

Theorem 3.2. *Assume that the following conditions are satisfied:*
1) $g(x)x < 0, \quad \forall x \in D = \{x \in \mathbb{R} : \ -H_1 \leq -H_0 \leq x < 0\}$*;*
2) condition 2 of Theorem 3.1 holds.
Then the solution $x = 0$ *of equation* (2.1) *is unstable.*

P r o o f. For definiteness, let us assume that condition 1 of the theorem holds for all $x \in D$. From this, using inequalities (3.1) one can find the functions $d_1, d_2 \in \mathcal{K}$ such as

$$d_1(|x|) \leq g(x) \leq L(H_1)|x|; \quad -G(x) \leq d_2(|x|) \quad \forall x \in D. \tag{3.5}$$

Consider the functional

$$V = V(t,\varphi) = -G(\varphi(0)) - \frac{1}{2}a_0 \int\limits_{-h(t)}^{0} g^2(\varphi(\tau))d\tau. \tag{3.6}$$

By virtue of inequalities (3.5), there exists an open set $\mathbb{C}_0$ with boundary $\{x = 0\} \in \partial\mathbb{C}_0$ defined by the equality $\mathbb{C}_0 = \{\varphi \in \mathbb{C} : \ \varphi(s) < 0, \ \|\varphi\| \leq H_0, \ V(t,\varphi) > 0\}$.

Moreover, by the definition (3.6) of $V(t,\varphi)$, we have

$$V = V(t,\varphi) \leq d_2(|\varphi(0)|) \quad \forall \varphi \in \mathbb{C}_0. \tag{3.7}$$

For the derivative of the functional $V = V(t,\varphi)$ by virtue of condition 2, in the domain $\mathbb{R}^+ \times \{\varphi \in \mathbb{C} : \ \varphi(s) \leq 0, \ \|\varphi\| < H_1\}$, we have the estimates

$$\dot{V}(t,\varphi) = -a(t,\varphi)g^2(\varphi(0)) - b(t,\varphi)g(\varphi(0))q(t,\varphi(-h(t))) - \frac{1}{2}a_0 g^2(\varphi(0))+$$

$$+\frac{1}{2}a_0g^2(\varphi(-h(t)))(1-\dot{h}(t)) \geq -(a(t,\varphi)+\frac{1}{2}a_0)g^2(\varphi(0))-$$

$$-a_0l(\varphi)(1-\dot{h}(t))^{1/2}g(\varphi(0))g(\varphi(-h(t)))+$$

$$+\frac{1}{2}a_0g^2(\varphi(-h(t)))(1-\dot{h}(t)) \geq -\frac{1}{2}(A(t,\varphi)+a_0)g^2(\varphi(0)) \geq$$

$$\geq \frac{1}{2}\varepsilon_0g^2(\varphi(0)) \geq \frac{1}{2}\varepsilon_0d_1^2(|\varphi(0)|). \tag{3.8}$$

From inequalities (3.7) and (3.8) along the solution $x = x(t,\alpha,\varphi)$, $(\alpha,\varphi) \in \mathbb{R}^+ \times \mathbb{C}_0$ of equation (2.1), we obtain the relation

$$\dot{V}(t,x_t(\alpha,\varphi)) \geq \frac{1}{2}\varepsilon_0d_1^2(d_2^{-1}(V(t,x_t(\alpha,\varphi)))).$$

Hence, it follows that $x_t(\alpha,\varphi) \in \mathbb{C}_0$ $\forall t \in [\alpha,\beta)$, and the equality $x(t,\alpha,\varphi) = -H_0$ holds for $t = \beta$. This proves the theorem.

Let us study the uniform attraction problem of the zero solution of equation (2.1). For this, note that T. A. Burton's theorem from [12] can be developed for such a problem as well. On this basis, similarly to Theorem 3.1, the following theorem can be proved.

Theorem 3.3. *Assume that inequality* (3.1) *and the conditions of Theorem 3.1 are satisfied for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ *and all* $(t,\varphi) \in \mathbb{R}^+ \times \mathbb{C}$*, and also the following holds*

3) there exist sequences $x_k \to \infty$ *and* $x_l \to -\infty$ *such that* $G(x_k) \to \infty$ *and* $G(x_l) \to \infty$ *as* $k,l \to \infty$.

Then the solutions of equation (2.1) *are uniformly bounded and the zero solution* $x = 0$ *of equation* (2.1) *is globally uniformly asymptotically stable.*

Example 3.1. Consider the equation

$$\dot{x}(t) = a(t)\sin x(t) + b(t)\sin(\mu(t)x(t-h(t))), \tag{3.9}$$

in which the delay $h(t)$ satisfies the inequalities (3.2), and the coefficients $a(t)$ and $b(t)$ are such that the following inequalities are satisfied

$$a(t) \leq -a_0 - \varepsilon \quad (a_0 > 0,\ \varepsilon > 0), \quad |b(t)\mu(t)| \leq a_0(1-\dot{h}(t))^{1/2}.$$

Based on Theorem 3.1, one can find that the solution $x = 0$ of the equation (3.9) is uniformly asymptotically stable. Moreover, $x = 0$ is the point of attraction for all solutions $x = x(t,\alpha,\varphi)$ bounded at $t \geq \alpha$ by the region $\{|x| \leq \frac{\pi}{2}\}$. It is easy to find an estimate of the area of attraction based on the functional

$$V(t,\varphi) = 1 - \cos(\varphi(0)) + \frac{1}{2}\int_{-h(t)}^{0} \sin^2(\varphi(\tau))d\tau.$$

When the condition regarding the coefficient $a(t)$ changes to the condition $a(t) \geq a_0 + \varepsilon$ $(a_0 > 0,\ \varepsilon > 0)$, the solution $x = 0$ to the equation (3.9) is unstable according to Theorem 3.2.

4. On the solution to the stability problem when the right-hand side of equation (2.1) is bounded

Let us consider the stability problem posed under the conditions of boundedness of the functions $a(t,\varphi)$ and $b(t,\varphi)$

$$|a(t,\varphi)| \le a_0 = const, \quad |b(t,\varphi)| \le b_0 = const$$

$$\forall (t,\varphi) \in \mathbb{R}^+ \times \{\varphi \in \mathbb{C}: \ \|\varphi\| \le H_1 = const > 0\}. \tag{4.1}$$

Consider the equation

$$\dot{y}(t) = a_1(t,y_t)g(y(t)) + b_1(t,y_t)q_1(t,y(t)) - b_1(t,y_t)\int\limits_{t-h_1(t)}^{t} a_1(\tau,y_\tau)\frac{\partial q_1(\tau,y(\tau))}{\partial y}g(y(\tau))d\tau -$$

$$-b_1(t,y_t)\int\limits_{t-h_1(t)}^{t} b_1(\tau,y_\tau)\frac{\partial q_1(\tau,y(\tau))}{\partial y}q_1(\tau,y(\tau-h_1(\tau)))d\tau, \tag{4.2}$$

$$a_1(t,\psi) = a(h_0+t,\psi), \quad b_1(t,\psi) = b(h_0+t,\psi)$$

$$q_1(t,y) = q(h_0+t,y), \quad h_1(t) = h(h_0+t)$$

with domain $\mathbb{R}^+ \times \mathbb{C}_\psi$, where $\mathbb{C}_\psi$ is the Banach space of continuous functions $\psi: \ [-2h_0, 0] \to \mathbb{R}$.

Each solution $x = x(t,\alpha,\varphi)$ $(\alpha,\varphi) \in \mathbb{R}^+ \times \mathbb{C}$ of equation (2.1) coincides at $t \ge \alpha + h_0$ with the solution $y = y(t,\alpha,\psi)$ of equation (4.2) with the initial function $\psi: \ [-2h_0, 0] \to \mathbb{R}$ defined by the equality

$$\psi(s) = \varphi(h_0+s), \quad -2h_0 \le s \le -h_0;$$

$$\psi(s) = x(\alpha+s+h_0,\alpha,\varphi), \quad -h_0 \le s \le 0.$$

Therefore, from the solutions limiting properties of equation (4.2), similar solutions limiting properties of equation (2.1) hold. In addition to conditions (4.1), we also assume that there exist the function $\mu_1 = \mu_1(t)$, $\mu_1 : \mathbb{R} \to \mathbb{R}$ and the positive reals μ_2 and M such as

$$\begin{gathered} b(t,\varphi) \le 0 \quad \forall (t,\varphi) \in \mathbb{R}^+ \times \{\|\varphi\| \le H_1\}, \\ 0 \le \mu_1(t) \le q(t,x)/g(x) \le \mu_2, \ \left|\frac{\partial q(t,x)}{\partial x}\right| \le M \quad \forall (t,x) \in \mathbb{R}^+ \times \{x \in \mathbb{R}: \ |x| \le H_1\}. \end{gathered} \tag{4.3}$$

Theorem 4.1. *Assume that the conditions are fulfilled:*
1) $g(x)x > 0, \quad \forall x \in \{x \in \mathbb{R}: \ 0 < |x| \le H_0 < H_1\}$*;*
2) $B(t,\varphi) = a(t,\varphi) + \mu_1(t)b(t,\varphi) + Mb_0h_0(a_0+\mu_2 b_0) \le -\varepsilon_0 < 0 \quad \forall (t,\varphi) \in \mathbb{R}^+ \times \{\varphi \in \mathbb{C} : \|\varphi\| \le H_1\}$.
Then the solution $x = 0$ *of equation* (2.1) *is uniformly asymptotically stable.*

P r o o f. As noted above, the solution to the stability problem for the zero solution $x = 0$ of equation (2.1) is reduced to the solution of the stability problem for the solution $y = 0$ of equation (4.2).

To solve the stability problem, we choose the Lyapunov functional candidate as follows

$$V_1(\psi) = G(\psi(0)) + \nu_1 \int\limits_{-h_0}^{0} \left(\int\limits_{\tau}^{0} g^2(\psi(s))ds \right) d\tau + \nu_2 \int\limits_{-2h_0}^{-h_0} \left(\int\limits_{\tau}^{0} g^2(\psi(s))ds \right) d\tau, \qquad (4.4)$$

where $\nu_1 = \frac{1}{2} M a_0 b_0$ and $\nu_2 = \frac{1}{2} M b_0^2$.

The derivative of the functional $V_1(\psi)$ by virtue of equation (4.2) has the following expression

$$\dot{V}_1(t,\psi) = a_1(t,\psi) g^2(\psi(0)) + b_1(t,\psi) q_1(t,\psi(0)) g(\psi(0)) -$$

$$-b_1(t,\psi)(\int\limits_{-h_1(t)}^{0} a_1(t+\tau,\psi_\tau) \frac{\partial q(t,x)}{\partial x}|_{x=\psi(\tau)} g(\psi(0)) g(\psi(\tau)) d\tau -$$

$$- \int\limits_{-h_1(t)}^{0} b_1(t+\tau) \frac{\partial q(t,x)}{\partial x}|_{x=\psi(\tau)} g(\psi(0)) q_1(\tau, \psi(\tau - h_1(\tau))) d\tau) +$$

$$+\nu_1 \int\limits_{-h_0}^{0} (g^2(\psi(0)) - g^2(\psi(\tau))) d\tau + \nu_2 \int\limits_{-2h_0}^{-h_0} (g^2(\psi(0)) - g^2(\psi(\tau))) d\tau. \qquad (4.5)$$

By virtue of conditions (4.3), the derivative (4.5) has the estimates

$$\dot{V}_1(t,\psi) \le (a_1(t,\psi) + \mu_1(t) b_1(t,\psi) + (\nu_1 + \nu_2) h_0) g^2(\psi(0)) +$$

$$+M|b_1(t,\psi)| \int\limits_{-h_0}^{0} |a_1(t+\tau,\psi)||g(\psi(0))||g(\psi(\tau))| d\tau +$$

$$+M\mu_2 |b_1(t,\psi)| \int\limits_{-2h_0}^{-h_0} |b_1(t+\tau,\psi)||g(\psi(0))||g(\psi(\tau))| d\tau \le$$

$$\le (a_1(t,\psi) + \mu_1(t) b_1(t,\psi) + M b_0 h_0 (a_0 + \mu_2 b_0)) g^2(\psi(0)) = B_1(t,\psi) g^2(\psi(0)). \qquad (4.6)$$

Thus, using (4.6) we find that

$$\dot{V}(t,\psi) \le -\varepsilon_0 g^2(\psi(0))$$

by virtue of condition 2 of the theorem.

According to the theorems from the monographs [3], [4], we obtain the proof of the theorem.

Assume that the function $g(x)$ satisfies the condition

$$|g(x)| \le L(H_1)|x| \quad \forall x \in \{x \in \mathbb{R} : \ |x| \le H_1 > 0\}.$$

Theorem 4.2. *Assume that condition 1) of Theorem 3.2 is satisfied as well as the following condition*

2) $A(t,\varphi) = a(t,\varphi) + b(t,\varphi) + M b_0 h_0 (a_0 + b_0) \le -\varepsilon_0 < 0$ *holds.*

Then the solution $x = 0$ *of equation* (2.1) *is unstable.*

P r o o f. To obtain the proof, as in Theorem 3.2, note that the relationship (3.5) holds. Consider the functional $V_2(\psi) = -V_1(\psi)$ in accordance with formula (4.4).

As in Theorem 3.2, we find the set $\mathbb{C}_0(V_2)$ as follows

$$\mathbb{C}_0(V_2) = \{\psi \in \mathbb{C}_\psi : \psi(s) < 0, \|\psi\| \leq H_0, V_2(\psi) > 0\}.$$

We have an inequality of type (3.7)

$$V_2(\psi) \leq d_2(|\psi(0)|) \quad \forall \psi \in \mathbb{C}_0(V_2).$$

For the derivative of the functional $V_2 = V_2(\psi)$ by virtue of condition 2 of the theorem, in the domain $\mathbb{R}^+ \times \{\psi \in \mathbb{C}_\psi : \psi(s) \leq 0, \|\psi\| \leq H_1\}$, in accordance with calculations (4.5) and (4.6), we find

$$\dot{V}_2(t, \psi) \geq \varepsilon_0 d_1^2(d_2^{-1}(V_2(\psi))).$$

From here, as in Theorem 3.2, we get the instability property of the solution $y = 0$ of equation (4.2) and, accordingly, the solution $x = 0$ of equation (2.1) is unstable. This completes the proof.

E x a m p l e 4.1. Consider the equation

$$\dot{x}(t) = a(t)x^3(t) + b(t)\sin^3(x(t - h(t))), \tag{4.7}$$

for which the following inequalities are satisfied

$$|a(t)| \leq a_0, \quad -b_0 \leq b(t) \leq 0, \quad 0 \leq h(t) \leq h_0 \quad \forall t \in \mathbb{R}^+. \tag{4.8}$$

Based on Theorem 4.1, one can find that if the following condition holds

$$a(t) + \frac{1}{8}b(t) + 3b_0h_0(a_0 + b_0) \leq -\varepsilon_0 < 0,$$

then the solution $x = 0$ of equation (4.7) is uniformly asymptotically stable with the attraction of all solutions $x = x(t, \alpha, \varphi)$ bounded at $t \geq \alpha$ by the region $\{x \in \mathbb{R} : |x| \leq \pi/2\}$.

Let us study the problem of the global behavior of solutions, when equation (2.1) has an infinite number of equilibrium positions, of course on a finite interval. The boundedness of the right-hand side of equation (2.1) by virtue of conditions (4.1) and (4.3) on bounded solutions ensures their uniform continuity as $t \in [\alpha + h, \infty)$. This allows us to apply the theorem on the solutions limiting behavior of functional-differential equations from [19], [20].

Let us consider the problem of the solutions attraction under the assumption

$$|q(t, x)| \leq \mu_0|g(x)| \quad (\mu_0 > 0) \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Denote $N_0 = \{x \in \mathbb{R} : g(x) = 0\}$. Assume that for each $H_1 > 0$ the set $N_0 \cap \{x \in \mathbb{R} : |x| \leq H_1 < \infty\}$ consists of a finite number of points.

T h e o r e m 4.3. *Assume that:*

1) dependencies $a(t, \varphi)$ and $b(t, \varphi)$ satisfy the inequality

$$B(t, \varphi) = a(t, \varphi) + \mu_0|b(t, \varphi)| + Mb_0h_0(a_0 + b_0\mu_0) \leq -\varepsilon_0 < 0 \quad \forall (t, \varphi) \in \mathbb{R}^+ \times \mathbb{C};$$

2) there exist sequences $x_k \to \infty$ and $x_l \to \infty$ such that $G(x_k) \to \infty$ and $G(x_l) \to \infty$ as $k, l \to \infty$.

Then each solution of equation (2.1) converges asymptotically to one of the equilibrium positions as $t \to \infty$, i. e. there exists the point $x^* \in N_0$ such as $x(t, \alpha, \varphi) \to x^*$ as $t \to \infty$.

P r o o f. As shown, the problem of the limiting behavior of solutions to equation (2.1) reduces to the same problem for equation (4.2).

Let $H_0 > 0$ be an arbitrary positive real, $\mathbb{C}_{0\psi} = \{\psi \in \mathbb{C}_\psi : \|\psi\| \leq H_0\}$. For the functional (4.4), we have the estimate

$$V_1(\psi) \leq G(\psi(0)) + \nu_1 \int_{-h_0}^{0} (\int_{\tau}^{0} g^2(\psi(s))ds)d\tau + \nu_2 \int_{-2h_0}^{-h_0} (\int_{\tau}^{0} g^2(\psi(s))ds)d\tau \leq V_0,$$

$$V_0 = sup_{|x|\leq H_0}(|g(x)|(H_0 + (\nu_1+\nu_2)h_0^2|g(x)|/2)).$$

According to condition 2 of the theorem, there is a value $H_1 > H_0$ such that $G(H_1) > V_0$. By virtue of condition 1 of the theorem, the derivative $\dot{V}_1(t,\psi)$ of the functional $V_1(\psi)$ satisfies the inequality

$$\dot{V}_1(t,\psi) \leq (a(t,\varphi) + \mu_0|b(t,\varphi)| + Mb_0h_0(a_0 + b_0\mu_0))g^2(\psi(0)) \leq -\varepsilon_0 g^2(\psi(0)) \leq 0.$$

For any solution $y = y(t,\alpha,\psi)$, $(\alpha,\psi) \in \mathbb{R}^+ \times \mathbb{C}_{0\psi}$ of equation (4.2), the inequality $V_1(y_t(\alpha,\psi)) \leq V_0 < G(H_1)$ holds, and hence, $|y(t,\alpha,\psi)| \leq H_1 \quad \forall t \geq \alpha$.

According to the theorem from [19], [20], it also follows that each solution of (4.2) is such that $y(t,\alpha,\psi) \to N_0$ as $t \to \infty$. But since the set $N_1 = N_0 \cap \{x \in \mathbb{R} : |x| \leq H_1\}$ consists of a finite number of points, it follows that there exists $x_0^* \in N_1$ such that $y(t,\alpha,\psi) \to x_0^*$ as $t \to \infty$ [19], [20].

The following corollary directly follows from Theorem 4.3.

C o n s e q u e n c e 4.1. *Assume that under the conditions of Theorem 4.3, also the inequality* $g(x)x > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$ *holds. Then the solution* $x = 0$ *of equation* (2.1) *is globally uniformly asymptotically stable.*

E x a m p l e 4.2. Consider the equation

$$\dot{x}(t) = a(t)x(t)\sin^2(x(t)) + b(t)\sin^3(x(t-h(t))), \tag{4.9}$$

in which the coefficients $a(t)$ and $b(t)$, and the delay function $h(t)$ satisfy the inequalities

$$a(t) \geq -a_0 \quad (a_0 > 0), \quad |b(t)| \leq b_0, \quad a(t) + |b(t)| + 3b_0h_0(a_0 + b_0) \leq -\varepsilon_0 < 0.$$

In accordance with Theorem 4.3, one cane find that the solution $x = 0$ of equation (4.9) is uniformly asymptotically stable. In addition, for each solution $x(t,\alpha,\varphi)$ of equation (4.9), the following holds $x(t,\alpha,\varphi) \to x_0^* \in N_1 = \{x \in \mathbb{R} : x = \pi k, \ k \in \mathbb{Z}\}$.

5. Conclusion

For a sufficiently wide class of non-linear non-autonomous scalar functional-differential equations, sufficient conditions for the limit behavior of their solutions are derived: stability, including robust and global uniform asymptotic stability of the zero solution; attraction of solutions to the equilibrium position. This class includes linear equations with variable delay and variable coefficients of a general form. The obtained results can be used to study stability based on the expansion of the right-hand side of various equations of this type, stability in the presence of perturbations.

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Submitted 08.09.2023; Revised 04.10.2023; Accepted 24.11.2023

The authors have read and approved the final manuscript.
Conflict of interest: The authors declare no conflict of interest.

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Поступила 08.09.2023; доработана после рецензирования 04.10.2023; принята к публикации 24.11.2023

Авторы прочитали и одобрили окончательный вариант рукописи.
Конфликт интересов: авторы заявляют об отсутствии конфликта интересов.