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On the Movement of Gyrostat under the Action of Potential and Gyroscopic Forces

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Abstract. A system of differential equations is considered that describes the motion of a gyrostat under the action of the moment of potential, gyroscopic and circular-gyroscopic forces. The form of the moment of forces is indicated for which the system has the three first integrals of a given form. An analog of V.I. Zubov's theorem for representing solutions of gyrostat equations by power series is given, and the possibility of using this approach to predict motions is shown. For an analogue of the Lagrange case, integration in quadratures is performed. Analogues of the case of full dynamical symmetry and the Hess case are also indicated. Based on the principle of optimal damping developed by V.I. Zubov, a design of the control moment created by circular-gyroscopic forces is proposed, which ensures that one of the coordinates reaches a constant (albeit unknown in advance) value or the transition of the state vector to the level surface of the particular Hess integral. A numerical example is given, for which a two-parameter family of exact almost periodic solutions, represented by trigonometric functions, is found.

Keywords: gyrostat, moment of potential and gyroscopic forces, first integrals, integrability, exact solutions, analogues of classical cases, control

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Оригинальная статья

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О движении гиростата под действием потенциальных и гироскопических сил

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Аннотация. Рассмотрена система дифференциальных уравнений, описывающая движение гиростата под действием момента потенциальных, гироскопических и циркулярно-гироскопических сил. Указан вид момента сил, при котором система имеет три первых интеграла заданного вида. Приводится аналог теоремы В. И. Зубова для представления решений уравнений гиростата степенными рядами и показана возможность применения такого подхода для прогнозирования движений. Для аналога случая Лагранжа производится интегрирование в квадратурах. Также указаны аналоги случая полной динамической симметрии и случая Гесса. На основе принципа оптимального демпфирования, разработанного В. И. Зубовым, предложена конструкция управляющего момента, создаваемого циркулярно-гироскопическими силами, обеспечивающая выход одной из координат на постоянную (хотя и неизвестную заранее) величину или переход вектора состояния на поверхность уровня частного интеграла Гесса. Приведен числовой пример, для которого найдено двухпараметрическое семейство точных почти периодических решений, представленных тригонометрическими функциями.

Ключевые слова: гиростат, момент потенциальных и гироскопических сил, первые интегралы, интегрируемость, точные решения, аналоги классических случаев, управление

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1. Introduction

The system of nonlinear differential equations describing the motion of a heavy solid body near a fixed point was obtained by L.Euler in the middle of the XVIII century and for a long time was an inspiring object for deep research of outstanding mathematicians and mechanics. Classical cases of integrability of this system have been found (cases of Euler, Lagrange and Kowalevski), for which there is an additional fourth algebraic first integral and it has been proved that if the conditions of these classical cases are not fulfilled, the additional integral does not exist even in the class of analytical functions (detailed history of research and overview of results can be found in monographs [1],[2],[3]), as well as in review article [4].

After the discovery of the Kowalevski integral, cases of existence of an additional particular integral were also found, when it is not possible to fully integrate equations, but it is possible to obtain separate private solutions [2]. Among them is the case of Hess, characterized by the existence of an additional linear particular integral, which was found for the equations of heavy solid motion in 1890 [1],[2]. Various issues related to the Hess case, its development and generalizations were examined in [5].

The classical Lagrange case for equations of motion of a heavy solid with a fixed point is highlighted by conditions of coincidence of two moments of inertia and linear dependence of a potential function on only one angle [1]. For the system discussed herein, it will also be necessary to impose additional conditions on the hyrostatic moment vector λ, the matrix S and the function $L(t, \gamma, \omega)$.

The classical case of full dynamic symmetry for the equations of motion of a heavy solid with a fixed point is highlighted by the conditions of coincidence of all three moments of inertia and linear dependence of the potential function on the angles of orientation, and the special choice of coordinate system is reduced to the Lagrange case [1]. For the system discussed in this article, which simulates the movement of hyrostate under the action of potential, gyroscopic and circular-gyroscopic forces, it will also be necessary to impose additional conditions on the vector of hyrostatic moment and parameters characterizing the moment created by gyroscopic and circular-gyroscopic forces. Therefore, a reduction to a case similar to the Lagrange case is not guaranteed by simply choosing a coordinate system here. Note that the case of complete dynamic symmetry is of interest and continues to be studied for the purpose of constructing solutions and integrals, for example in [6] for a solid with a spherical ellipsoid of inertia and constant moment, exact analytical solutions have been obtained.

2. Motion Equations and First Integrals

Consider the vector form of the equations of motion of a hyrostate with a fixed point under the influence of the moment of forces

$$I\dot{\omega} = (I\omega + \lambda) \times \omega + M, \tag{2.1}$$

$$\dot{\gamma} = \gamma \times \omega. \tag{2.2}$$

Here $\omega = \mathrm{col}(p, q, r)$ — the angular velocity vector, $\gamma = \mathrm{col}(\gamma_1, \gamma_2, \gamma_3)$ — the unit vector of the symmetry axis of the force field, given by projections on the axis of the associated coordinate system, $I = I^T > 0$ — the symmetric positive – definite matrix of the inertia tensor relative to a fixed point, $\lambda = \mathrm{col}(\lambda_1, \lambda_2, \lambda_3)$ — the gyrostatic moment vector, $M = M(t, \gamma, \omega)$ — the moment vector of forces acting on gyrostat. Following [7],[8],[9],[10], we will consider the following functions and relations as the first integrals

$$J_1 = J_1(\gamma, \omega) = \omega^T I\omega + 2U(\gamma) = d_1 = \mathrm{const}, \tag{2.3}$$

$$J_2 = J_2(\gamma, \omega) = \gamma^T (I\omega + \lambda) + \frac{1}{2}\gamma^T S\gamma = d_2 = \text{const}, \tag{2.4}$$

$$J_3 = J_3(\gamma) = \gamma^T \gamma = 1. \tag{2.5}$$

where $S = S^T$ is some symmetric matrix. The following assertion was proved in [11].

Theorem 2.1. *In order for the functions* (2.3)–(2.5) *to be the first integrals for the system* (2.1), (2.2) *it is necessary and sufficient for the moment* M *to be represented as*

$$M = \gamma \times \frac{\partial U}{\partial \gamma} - \omega \times S\gamma + L(t, \gamma, \omega)\omega \times \gamma, \tag{2.6}$$

where $L(t, \gamma, \omega)$ *is an arbitrary function.*

This statement shows that the first integrals (2.3) and (2.4) determine the moment M in the right part of (2.1) in a unique way up to the circular-gyroscopic component $L(t, \gamma, \omega)\omega \times \gamma$. The first term in formula (2.6) is the moment of potential forces, and the second is the moment of gyroscopic forces.

Next, we will consider the inertia matrix diagonal $I = \text{diag}(A, B, C)$. Let's write the system (2.1), (2.2) and the first integrals in coordinate form

$$\begin{gathered} A\dot{p} = (B - C)qr + \lambda_2 r - \lambda_3 q + \gamma_2 \frac{\partial U}{\partial \gamma_3} - \gamma_3 \frac{\partial U}{\partial \gamma_2} - q(S\gamma)_3 + r(S\gamma)_2 + L(q\gamma_3 - r\gamma_2), \\ B\dot{q} = (C - A)pr + \lambda_3 p - \lambda_1 r + \gamma_3 \frac{\partial U}{\partial \gamma_1} - \gamma_1 \frac{\partial U}{\partial \gamma_3} - r(S\gamma)_1 + p(S\gamma)_3 + \\ + L(r\gamma_1 - p\gamma_3), \\ C\dot{r} = (A - B)pq + \lambda_1 q - \lambda_2 p + \gamma_1 \frac{\partial U}{\partial \gamma_2} - \gamma_2 \frac{\partial U}{\partial \gamma_1} - p(S\gamma)_2 + q(S\gamma)_1 + L(p\gamma_2 - q\gamma_1), \end{gathered} \tag{2.7}$$

$$\dot{\gamma_1} = r\gamma_2 - q\gamma_3, \quad \dot{\gamma_2} = p\gamma_3 - r\gamma_1, \quad \dot{\gamma_3} = q\gamma_1 - p\gamma_2, \tag{2.8}$$

$$J_1 = Ap^2 + Bq^2 + Cr^2 + 2U(\gamma) = d_1 = \text{const}, \tag{2.9}$$

$$\begin{gathered} J_2 = \gamma_1(Ap + \lambda_1) + \gamma_2(Bq + \lambda_2) + \gamma_3(Cr + \lambda_3) + \\ + \frac{1}{2}\left(s_{11}\gamma_1^2 + s_{22}\gamma_2^2 + s_{33}\gamma_3^2\right) + s_{12}\gamma_1\gamma_2 + s_{13}\gamma_1\gamma_3 + s_{23}\gamma_2\gamma_3 = d_2 = \text{const}, \end{gathered} \tag{2.10}$$

$$J_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{2.11}$$

Here $(S\gamma)_i$ means the i-th component of the vector $S\gamma$. The right parts of the system (2.7), (2.8) are independent of those variables whose derivatives are present in the left parts, so Jacobi 's integration theory applies to this system. Where there is an additional independent of (2.9)–(2.11) the first integral, this system is integrable.

The main purpose of this article is to use an analogy with classical integrability cases for heavy solid equations to identify cases of the existence of an additional first integral and perform integration of the system (9), (10). This problem is considered and solved for the analog of the Lagrange case. Analogs for the case of complete dynamic symmetry and the Hess case are also identified, for which the conditions for the existence of General and partial integrals are obtained, respectively.

3. Analog of Zubov's theorem on analytical solutions

For the equations of motion of a heavy solid described by the system (2.7), (2.8) in the case of $S = 0$, $\lambda = 0$, $L(t, \gamma, \omega) \equiv 0$ and the linear function $U(\gamma)$, V. I. Zubov proved [3] the theorem that all real solutions are defined on the entire real axis and are represented by power series converging also on the entire real axis. The proof is essentially based on the properties of boundedness of all solutions and uniform analyticity of the right parts of the system of differential equations under consideration. Having provided these properties, we come to the validity of the analog of Zubov's theorem for the gyrostat equations.

Theorem 3.1. *If the functions $U(\gamma)$ and $L = L(\gamma, \omega)$ are uniformly analytic in each bounded region of the phase space (γ, ω), then all solutions of the system (2.7), (2.8) are defined and bounded on the entire real axis $t \in (-\infty, +\infty)$ and are represented by power series convergent for all $t \in (-\infty, +\infty)$*

$$p(t) = \sum_{k=0}^{+\infty} p_k \psi^k, \quad q(t) = \sum_{k=0}^{+\infty} q_k \psi^k, \quad r(t) = \sum_{k=0}^{+\infty} r_k \psi^k,$$

$$\gamma_1(t) = \sum_{k=0}^{+\infty} \gamma_{1k} \psi^k, \quad \gamma_2(t) = \sum_{k=0}^{+\infty} \gamma_{2k} \psi^k, \quad \gamma_3(t) = \sum_{k=0}^{+\infty} \gamma_{3k} \psi^k.$$

Here $\psi = \dfrac{\exp(\mu t) - 1}{\exp(\mu t) + 1}$, $\mu = \pi/2h$, $h > 0$ *— is some constant.*

For practical construction of series representing solutions, analytical computing systems can be successfully used. For qualitative analysis of the properties of solutions (for example, stability) over infinite or sufficiently long time intervals, such series are not applicable. However, they can be very useful for predicting movement over short time intervals if it is possible to measure the current state vector $(\gamma(t), \omega(t))$. In this case, the depth of forecasting for the future $\xi \in (t, t + \tau)$, $\tau > 0$ can be estimated fairly accurately by comparing for the segment $\xi \in (t - \tau, t)$ of the constructed power series with the solution $(\gamma(\xi), \omega(\xi))$ already known at the moment $t \in (-\infty, +\infty)$.

4. Analog to the Lagrange case

The classical Lagrange case for equations of motion of a heavy solid body with a fixed point is distinguished by the conditions of coincidence of two moments of inertia $B = A$ and the linear dependence of the potential function on only one angle $U(\gamma) = k\gamma_3$ [1]. For the system (2.7), (2.8) considered here, it is also necessary to impose additional conditions on the gyrostatic moment vector λ, the matrix S, and the function $L(t, \gamma, \omega)$. The following statement is true.

Theorem 4.1. *Let the following conditions be met for the system (2.7), (2.8):*

1. $B = A$, $\lambda_1 = \lambda_2 = 0$.
2. *Function $L(t, \gamma, \omega)$ is constant, i.e.* $L(t, \gamma, \omega) = L = \text{const}$.
3. *Matrix S is diagonal* $S = \text{diag}(s_{11}, s_{22}, s_{33})$, *with* $s_{22} = s_{11}$.

4. Function $U(\gamma)$ is an arbitrary continuously differentiable function of two arguments $U(\gamma) = U(\sigma, \gamma_3)$, where $\sigma = \gamma_1^2 + \gamma_2^2$.

Then the system (2.7), (2.8) *has in addition to integrals* (2.9)–(2.11) *an additional first integral*

$$J_4 = Cr + (L - s_{11})\gamma_3 = d_4 = \text{const} \qquad (4.1)$$

and is integrated in quadratures.

R e m a r k 4.1. *If $L = s_{11} + f(\gamma_3)$ where $f(\gamma_3)$ is some continuous function and all other conditions of statement 4.1 are met, then the additional integral instead of (4.1) has the form $J_4 = Cr + \int\limits_0^{\gamma_3} f(y)dy$ and statement 4.1 remains valid.*

5. Analog of the case of complete dynamic symmetry

The classical case of complete dynamic symmetry for equations of motion of a heavy solid with a fixed point is distinguished by the conditions of coincidence of all three moments of inertia $A = B = C$ and the linear dependence of the potential function $U(\gamma)$ on the angles [1]. For the system(2.7), (2.8) considered here, it is also necessary to impose additional conditions on the gyrostatic moment vector λ, the matrix S, and the function $L(t, \gamma, \omega)$. The following assertion was proved in [11].

T h e o r e m 5.1. *Let the following conditions be met for the* (2.7), (2.8)*:*

1. *$A = B = C$.*

2. *Function $L(t, \gamma, \omega)$ is constant, i.e. $L(t, \gamma, \omega) = L = \text{const}$.*

3. *Matrix S has the form*

$$S = \zeta \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

where ζ, a, b, c are arbitrary constants.

4. *Function $U(\gamma)$ has the form $U(\gamma) = F(a\gamma_1 + b\gamma_2 + c\gamma_3)$, where $F(\theta)$ an arbitrary continuously differentiable function of a single argument $\theta = a\gamma_1 + b\gamma_2 + c\gamma_3$.*

5. *The components of the gyrostatic moment vector $\lambda = \text{col}\,(\lambda_1, \lambda_2, \lambda_3)$ satisfy the linear system*

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then the system (2.7), (2.8) *has in addition to integrals* (2.9)–(2.11) *an additional first integral*

$$J_4 = A(ap + bq + cr) + L\,(a\gamma_1 + b\gamma_2 + c\gamma_3) = d_4 = \text{const}$$

and is integrated in quadratures.

It should be noted that in the classical case of complete dynamic symmetry for the equations of motion of a heavy solid with a fixed point [1], we have $L = 0$, $\lambda = 0$, $S = 0$, $U = a\gamma_1 + b\gamma_2 + c\gamma_3$ and a special choice of the associated coordinate system reduces the problem to the Lagrange case. However, for non-zero λ, S, satisfying the conditions of statement 5.1, such a coordinate replacement does not, in general, guarantee the fulfillment of the conditions of statement 4.1 for the system in the new coordinates.

6. Analog to the Hess case

The Hess case, characterized by the existence of an additional fourth linear partial integral, was found for the equations of motion of a heavy solid in 1890 [1],[2]. Review of further research related to this case and its analogues, as well as new results, is given in [5]. Here we give the conditions for the existence of a partial integral for the system (2.7), (2.8). The following statement is true.

T h e o r e m 6.1. *Let the system* (2.7), (2.8) *meet the conditions:*

1. *Matrix S has the form*

$$S = \begin{pmatrix} s_{11} & 0 & s_{13} \\ 0 & 0 & 0 \\ s_{13} & 0 & s_{33} \end{pmatrix}, \quad \text{and besides} \quad s_{13}^2 = s_{11}s_{33} \neq 0.$$

2. $s_{13}^2 C(A - B) = s_{11}^2 A(B - C)$.

3. *Components of the gyrostatic moment vector $\lambda = \mathrm{col}\,(\lambda_1, \lambda_2, \lambda_3)$ satisfy the following equations $\lambda_2 = 0$, $s_{13}\lambda_1 - s_{11}\lambda_3 = 0$.*

4. *Function $U(\gamma)$ has the form $U(\gamma) = F(s_{11}\gamma_1 + s_{13}\gamma_3)$, where $F(\theta)$ an arbitrary continuously differentiable function of a single argument $\theta = s_{11}\gamma_1 + s_{13}\gamma_3$.*

5. *Function $L(t, \gamma, \omega)$ is represented as a product*

$$L = L_1\,(s_{11}Ap + s_{13}Cr)\,L_2(t, \gamma, \omega),$$

where $L_1(0) = 0$, and function $L_2(t, \gamma, \omega)$ is arbitrary.

Then the system (2.7), (2.8) *has in addition to the integrals* (2.9)–(2.11) *an linear partial integral*

$$J_4 = s_{11}Ap + s_{13}Cr = 0.$$

7. Case of the control moment created by circular-gyroscopic forces

V. I. Zubov successfully applied the principle of optimal damping to solve problems of rotational motion control, using kinetic energy as the damped function [3]. Following this principle, we will now consider in (2.7) the moment $L(t, \gamma, \omega)\omega \times \gamma$, created by circularly-gyroscopic forces, as a control, i.e. the function $L(t, \gamma, \omega)$ can be selected to achieve certain goals. Since, according to statement 2.1 , the system (2.7), (2.8) will have the three first integrals (2.9)–(2.11) , the control goals can only be very limited. For example, this control cannot provide asymptotic stability of any solution, or control from an arbitrary given initial state to an arbitrary final state. However, in some cases, by selecting the function $L(t,\gamma,\omega)$ it is possible to achieve local control goals, for example, to output one of the values of $\gamma_i(t)$ to a constant (although unknown in advance) value, or to transfer the state vector to the surface of the level of the Hess partial integral.

Let's assume that the control function can take bounded values $|L(t,\gamma,\omega)| \le L_0 < +\infty$ and choose it so as to provide optimal damping of the function $V_1(p,q) = A^2Bp^2 + B^2Aq^2$.

Theorem 7.1. *Let the system* (2.7), (2.8) *meet the conditions:*

1. *$B = A$, $\lambda_1 = \lambda_2 = 0$.*
2. *Matrix S is diagonal $S = \mathrm{diag}(s_{11}, s_{22}, s_{33})$, with $s_{22} = s_{11}$.*
3. *Function $U(\gamma) \equiv 0$.*
4. *Control moment in the system* (2.7), (2.8) *is selected as $L(t,\gamma,\omega)\omega \times \gamma$, where $L = -L_0|r|\mathrm{sign}(q\gamma_1 - p\gamma_2)$, with $L_0 > |s_{11}|$.*

Then for each solution of the system (2.7), (2.8)*, the function $V_1(p,q)$ decreases to a constant value, and the component of the solution $\gamma_3(t)$ reaches a constant value in a finite time.*

Now we will choose the control function $L(t,\gamma,\omega)$ so as to ensure optimal damping of the function $V_2(p,r) = s_{11}Ap + s_{13}Cr$ to zero.

Theorem 7.2. *Let the system* (2.7), (2.8) *meet the conditions* 1–4 *of statement 6.1, and the control moment in the system is chosen as $L(t,\gamma,\omega)\omega \times \gamma$, where*

$$L = -L_0 \mathrm{sign}\left(s_{11}Ap + s_{13}Cr\right) \mathrm{sign}\left(s_{11}(q\gamma_3 - r\gamma_2) + s_{13}(p\gamma_2 - q\gamma_1)\right),$$

and $L_0 > 0$ is a sufficiently large number. Then each solution of the system (2.7), (2.8) *in a finite time reaches the set of the level of the Hess partial integral $V_2(p,r) = s_{11}Ap + s_{13}Cr = 0$.*

8. Example

Family of exact almost periodic solutions. Consider the following parameter values $A = B = 1$, $C = \frac{3}{2}$, $L = 1$, $s_{11} = s_{22} = s_{33} = 1$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -1$ and a potential function $U = -\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)$. Using statement 4.1, for the values of the first integrals $J_1 = 1$, $J_2 = -1$, $J_4 = -1$ we obtain a parametric family of almost periodic solutions

$$p(t) = \frac{\sqrt{21}}{3}\cos\varphi(t), \quad q(t) = \frac{\sqrt{21}}{3}\sin\varphi(t), \quad r(t) = -\frac{2}{3},$$

$$\gamma_1(t) = -\frac{7\sqrt{19}}{38}\sin\varphi(t)\cos\psi(t) - \frac{3\sqrt{21}}{38}\cos\varphi(t) + \frac{7\sqrt{3}}{19}\cos\varphi(t)\sin\psi(t),$$

$$\gamma_2(t) = \frac{7\sqrt{19}}{38}\cos\varphi(t)\cos\psi(t) - \frac{3\sqrt{21}}{38}\sin\varphi(t) + \frac{7\sqrt{3}}{19}\sin\varphi(t)\sin\psi(t),$$

$$\gamma_3(t) = \frac{\sqrt{7}}{266}\left(18\sqrt{7} + 49\sin\psi(t)\right),$$

where $\varphi(t) = C_2 - \dfrac{4}{3}t$, $\psi(t) = \dfrac{\sqrt{57}}{3}(C_1 - t)$. Here C_1, C_2 — arbitrary constants. Note that all solutions included in this family are combinations of harmonic oscillations with two incommensurable periods $T_1 = \dfrac{3\pi}{2}$, $T_2 = \dfrac{2\pi\sqrt{57}}{19}$.

Using the statement 3.1, using the Maple analytical computing system, we obtain a representation of solutions in the form of power series that coincide with the decompositions of the almost periodic solutions given above.

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