DOI 10.15507/2079-6900.22.202002.155-163

MSC2020 65M32

On an iterative method for solution of direct problem for nonlinear hyperbolic differential equations

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Abstract. An iterative method for solution of Cauchy problem for one-dimensional nonlinear hyperbolic differential equation is proposed in this paper. The method is based on continuous method for solution of nonlinear operator equations. The keystone idea of the method consists in transition from the original problem to a nonlinear integral equation and its successive solution via construction of an auxiliary system of nonlinear differential equations that can be solved with the help of different numerical methods. The result is presented as a mesh function that consists of approximate values of the solution of stated problem and is constructed on a uniform mesh in a bounded domain of two-dimensional space. The advantages of the method are its simplicity and also its universality in the sense that the method can be applied for solving problems with a wide range of nonlinearities. Finally it should be mentioned that one of the important advantages of the proposed method is its stability to perturbations of initial data that is substantiated by methods for analysis of stability of solutions of systems of ordinary differential equations. Solving several model problems shows effectiveness of the proposed method.

Key words: nonlinear hyperbolic equation, nonlinear integral equation, continuous operator method, system of differential equations, stability theory

1. Introduction

Hyperbolic differential equations are a powerful instrument of mathematical modelling of different processes and phenomena which modern science collides with. Among numerous applications of hyperbolic equations we can mention acoustics, elasticity theory, field theory, aerodynamics, electrodynamics etc. At present time there are many results in the field of development of numerical methods for solution of linear and nonlinear hyperbolic equations [1–7]. A wide range of numerical methods is used for approximate solution of hyperbolic equations:

- difference methods [6; 8],
- variational and projective methods [9],
- spline-collocation methods [10],
- approximate analytical methods [11],
- finite elements methods, factorization methods, decomposition methods [7; 12],
- boundary integral equation method

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and many others.

Nevertheless, until present time there is considerable demand for further development of methods for approximate solution of such equations: there is still the important problem of devising sufficiently precise and stable numerical methods that allow to solve hyperbolic equations effectively.

The numerical method that is proposed in this paper can be briefly described in the following way. By means of the general solution formula the original problem is replaced by a nonlinear integral equation. This equation is then solved with the help of continuous operator method. For this purpose a parametric differential equation of special type is composed and then solved approximately by means of one of methods for approximate solution of differential equations. As a result of solution we have a set of approximate values of the solution of original Cauchy problem on a uniform mesh.

Let us introduce a brief description of continuous operator method following the paper [13]. Suppose it is required to find a solution of the operator equation Let us introduce a brief description of continuous operator method following the paper [13]. Suppose it is required to find a solution of the operator equation

$$\Phi(\mathbf{y}) = \mathbf{f}, \tag{1.1}$$

where $\Phi : \mathrm{B} \to \mathrm{B}$, $\mathbf{y} \in \mathrm{B}$, $\mathbf{f} \in \mathrm{B}$, B — a Banach space. The equation (1.1) is placed in correspondence with the Cauchy problem for the following auxiliary parametric differential equation

$$\frac{\mathrm{d}\overline{\mathbf{y}}}{\mathrm{d}\sigma} = \Phi\left(\overline{\mathbf{y}}(\sigma)\right) - \mathbf{f}, \tag{1.2}$$

$$\overline{\mathbf{y}}(0) = \chi, \tag{1.3}$$

where $\sigma \geqslant 0$ and χ — arbitrary element of a Banach space B.

Let $\Lambda(\Phi)$ a logarithmic norm of the operator Φ.

R e m a r k 1.1 *Recall that a logarithmic norm $\Lambda(\Phi)$ of the operator Φ in a Banach space* B *is defined by the formula*

$$\Lambda(\Phi) = \lim_{h \downarrow 0} \frac{\|I + h\Phi\| - 1}{h},$$

where the symbol $h \downarrow 0$ denotes decreasing convergence to zero.

The following theorem holds [13].

T h e o r e m 1.1 *Suppose the equation* (1.1) *has the solution* $\mathbf{y}^*$, *and on every differentiable curve $g(s)$ located in a ball* $\mathrm{R}(\mathbf{y}^*, r)$ *the following conditions hold*

1. *For every s ($s > 0$):* $\displaystyle\int_0^s \Lambda\left(\Phi'(g(\tau))\right) d\tau \leqslant 0.$

2. $\displaystyle\lim_{s\to\infty} \left\{ \frac{1}{s} \int_0^s \Lambda\left(\Phi'(g(\tau))\right) d\tau \right\} \leqslant \gamma_g, \quad \gamma_g > 0.$

Then a solution of the problem (1.2)–(1.3) *converges to the solution* $\mathbf{y}^*$ *of the equation* (1.1).

2. Description of the method

Consider the Cauchy problem for a nonlinear hyperbolic differential equation:

$$\frac{\partial^2 u}{\partial t^2} = a\,\frac{\partial^2 u}{\partial x^2} + \Psi\left(t, x; u\right), \tag{2.1}$$

$$u(0, x) = \alpha(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \beta(x). \tag{2.2}$$

Here $0 \leqslant t \leqslant T$, $x \in \mathbb{R}$. A value $a > 0$ and functions $\alpha(x)$, $\beta(x)$ are supposed to be known.

Let us state the problem of approximate recovering of a function $u(t, x)$ that is the solution of the problem (2.1)–(2.2), in a finite set of points denoted as Ω.

R e m a r k 2.1 *Parameters of the set Ω will be determined below.*

Suppose that functions $\alpha(x)$, $\beta(x)$ are known in terms of finite sets of values (precise or approximate)

$$\{\alpha_0,\ \alpha_1,\ \ldots\ ,\ \alpha_N\} \qquad (\alpha_l = \alpha(x_l))$$
$$\{\beta_0,\ \beta_1,\ \ldots\ ,\ \beta_N\} \qquad (\beta_l = \beta(x_l))\,,$$

where $x_l = -A + lh$, $l = \overline{0, N}$, $h = 2A/N$ — fixed step with respect to x variable.

It is known [14] that the problem (2.1)–(2.2) reduces to the problem of solving the following integral equation

$$u(t, x) = \frac{1}{2}\left[\alpha(x - at) + \alpha(x + at)\right] + \frac{1}{2a}\int\limits_{x-at}^{x+at} \beta(\xi)\, d\xi + $$
$$+ \frac{1}{2a}\int\limits_{0}^{t}\int\limits_{x-a(t-s)}^{x+a(t-s)} \Psi\left(s, \xi, u(s, \xi)\right)\, d\xi\, ds. \tag{2.3}$$

A description and a realization of the proposed method could be significantly simplified with the auxiliary constraint that sets a relation between steps of a mesh. Specifically, let τ (the step with respect to t) be specified in such a way that the following condition holds.

$$\tau = \frac{h}{a}. \tag{2.4}$$

Let us define a positive integer number M so as to fulfill the condition

$$0 \leqslant \frac{\widetilde{T} - T}{\tau} < 1,$$

where $\widetilde{T} = M\tau$. Let us also denote $u_{i,j} = u(t_i, x_j)$ and write the equation (2.3) at some fixed point (t_i, x_j).

$$u_{i,j} = \frac{1}{2}\left[\alpha(x_j - at_i) + \alpha(x_j + at_i)\right] + \frac{1}{2a}\int\limits_{x_j-at_i}^{x_j+at_i} \beta(\xi)\, d\xi + $$
$$+ \frac{1}{2a}\int\limits_{0}^{t_i}\int\limits_{x_j-a(t_i-s)}^{x_j+a(t_i-s)} \Phi\left(s, \xi, u(s, \xi)\right)\, d\xi\, ds. \tag{2.5}$$

It is easily seen that due to the relation (2.4) we have

$$x_j \pm at_i = -A + jh \pm ih = -A + (j \pm i)h. \tag{2.6}$$

Thus for every ordered couple of indices (i, j), where $i = \overline{1, M}$ and $j = \overline{i, N-i}$, there are such values $m \in \{\overline{0, N}\}$, $n \in \{\overline{0, N}\}$ that $x_j + at_i = x_m$ and $x_j - at_i = x_n$.

Hence we define the mentioned set Ω as follows.

$$\Omega = \left\{(t_i, x_j) : \quad i = \overline{1, M},\ j = \overline{i, N-i}\right\}. \tag{2.7}$$

Denote as $\varphi_{i,j}$ the approximation with the help of one of quadrature formulae of the integral function

$$\frac{1}{2}\left[\alpha(x_j - at_i) + \alpha(x_j + at_i)\right] + \frac{1}{2a}\int\limits_{x_j - at_i}^{x_j + at_i} \beta(\xi)\, d\xi.$$

In order to approximate the integral $\frac{1}{2a}\int\limits_0^{t_i}\int\limits_{x_j - a(t_i - s)}^{x_j + a(t_i - s)} \Psi\left(s, \xi, u(s, \xi)\right) d\xi\, ds$ we use compound trapezoid formula

$$\frac{1}{2a}\int\limits_0^{t_i}\int\limits_{x_j - a(t_i - s)}^{x_j + a(t_i - s)} \Phi\left(s, \xi, u(s, \xi)\right) d\xi\, ds \approx$$

$$\approx \frac{h^2}{16a} \cdot \sum_{k=0}^{i-1}\sum_{l=j-(i-k)}^{j+(i-k)-1} \left\{\Psi\left(t_k, x_l, u_{k,l}\right) + \Psi\left(t_k, x_{l+1}, u_{k,l+1}\right) + \right.$$
$$\left. + \Psi\left(t_{k+1}, x_l, u_{k+1,l}\right) + \Psi\left(t_{k+1}, x_{l+1}, u_{k+1,l+1}\right)\right\}. \tag{2.8}$$

Thus the equation (2.5) is approximated by the equation

$$u_{i,j} = \varphi_{i,j} + \frac{h^2}{16a} \cdot \sum_{k=0}^{i-1}\sum_{l=j-(i-k)}^{j+(i-k)-1} \left\{\Psi\left(t_k, x_l, u_{k,l}\right) + \Psi\left(t_k, x_{l+1}, u_{k,l+1}\right) + \right.$$
$$\left. + \Psi\left(t_{k+1}, x_l, u_{k+1,l}\right) + \Psi\left(t_{k+1}, x_{l+1}, u_{k+1,l+1}\right)\right\}, \tag{2.9}$$

where $i = \overline{1, M}$, $j = \overline{i, N-i}$.

The equation (2.9) forms the basis of the proposed iterative method for approximate solution of the problem (2.1)–(2.2).

Let us introduce functions $\overline{u}_{i,j}(\sigma)$, $\sigma \geqslant 0$ associated with corresponding unknown values $u_{i,j}$ through the limitary equation

$$\lim_{\sigma \to \infty} \overline{u}_{i,j}(\sigma) = u_{i,j}.$$

The auxiliary functions $\overline{u}_{i,j}(\sigma)$ form the solution of system of differential equations

$$\frac{d\overline{u}_{i,j}(\sigma)}{d\sigma} = \gamma_{i,j}\left\{\varphi_{i,j} + \frac{h^2}{16a} \cdot \sum_{k=0}^{i-1}\sum_{l=j-(i-k)}^{j+(i-k)-1} \left\{\Phi(t_k, x_l, \overline{u}_{k,l}(\sigma)) + \Phi(t_k, x_l, \overline{u}_{k,l+1}(\sigma)) + \right.\right.$$
$$\left.\left. + \Phi(t_k, x_l, \overline{u}_{k+1,l}(\sigma)) + \Phi(t_{k+1}, x_{l+1}, \overline{u}_{k+1,l+1}(\sigma))\right\}\right\}, \tag{2.10}$$

where $i = \overline{1, M}$, $j = \overline{i, N-i}$ and constant values $\gamma_{i,j}$ take on a value of $+1$ or -1.

R e m a r k 2.2 *The $\gamma_{i,j}$ values must be defined in such a way that the conditions of the theorem 1.1 are fulfilled and, as a consequence, iterative process based on the equation* (2.10) *converges. In practice values of these constants could be determined experimentally.*

In order to provide uniqueness of solution of the system (2.10), we must attach to it a set of initial conditions. It is known [13] that these conditions can be defined with arbitrary numbers. In particular, we can set all conditions equal to some fixed number. Besides — for the purpose of reducing the amount of computations — it is reasonable to define the initial conditions with the following formula:

$$\overline{u}_{i,j}(0) = \overline{u}_{i-1,j}(\sigma_L). \tag{2.11}$$

R e m a r k 2.3 *The formula* (2.11) *is applicable for $i = \overline{2, M}$. As values $u_{0,j}$ are not defined with continuous operator method but are known from initial conditions then for $i = 1$ the formula* (2.11) *is replaced by the formula*

$$\overline{u}_{1,j}(0) = \alpha(x_j). \tag{2.12}$$

For solving the system of equations (2.10) with the initial conditions (2.11)–(2.12) we can apply wide range of numerical methods for solving differential equations. In particular, such simple method as Euler method. We next describe its application.

Let θ — step of Euler method and L — number of iterations of Euler method. Denote $\sigma_r = r\theta$. Then the iterative process for solving the problem (2.10)–(2.11) is described by the formula

$$\overline{u}_{i,j,r} = \overline{u}_{i,j,k} + \gamma_{i.j}\Bigg\{\theta \cdot \{\varphi_{i,j} + \\ + \frac{h^2}{16} \cdot \sum_{k=0}^{i-1} \sum_{l=j-(i-k)}^{j+(i-k)-1} \{\Phi(t_k, x_l, \overline{u}_{k,l,r-1}) + \Phi(t_k, x_l, \overline{u}_{k,l+1,r-1}) + \\ + \Phi(t_k, x_l, \overline{u}_{k+1,l,r-1}) + \Phi(t_{k+1}, x_{l+1}, \overline{u}_{k+1,l+1,r-1})\}\}\Bigg\}, \tag{2.13}$$

where $\overline{u}_{i,j,r} = \overline{u}$ and $r = \overline{1, L}$.

The process (2.13) results in a set of values $\overline{u}_{i,j,L}$:

$$u_{i,j} \approx \overline{u}_{i,j,L}. \tag{2.14}$$

Thus the process of approximate solution of the problem (2.1)–(2.2) consists of the following steps.

1. The main parameters of the algorithm such as M, L and θ, are defined.
2. For $i = \overline{1, M}$, $j = \overline{i, N-i}$ values $\varphi_{i,j}$ are computed.
3. For $i = \overline{1, N}$ the following steps are performed successively.
   (a) For each $j = \overline{i, N-i}$ in accordance with the formula (2.11) initial values are $\overline{u}_{i,j}(0)$ defined while it is supposed that $\overline{u}_{0,j}(\sigma_L) = \alpha_j$.

(b) Using the formula (2.12) we perform L iterations of Euler method; in this connection r-th iteration of the method involves successive computing of the values $\overline{u}_{i,j,r}$ for $j = \overline{i, N-i}$.

(c) Results of the iterative process for $j = \overline{i, N-i}$ are defined with the formula (2.14).

(d) We make transition to the next value of i (next layer with respect to t) and return to the step (a).

Consider absolutely and irrespectively the way of constructing the approximation of $\Psi(t, x; u)$. In case Φ depends only on $u(t,x)$ construction of such approximation is straightforward. But in case Φ depends on the derivative $\dfrac{\partial u}{\partial x}$ the question arises about the manner in which this derivative should be approximated at the point (t_i, x_j) using only a set of values $\{u_{i,j}\}_{j=\overline{0,N}}$. The simplest way to do that is using difference approximations:

$$\left.\frac{\partial u}{\partial x}\right|_{t=t_i,\ x=x_j} = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{h} & \text{if} \quad j = 0, \\ \dfrac{u_{i,j+1} - u_{i,j-1}}{2h} & \text{if} \quad 1 \leqslant j \leqslant N-1, \\ \dfrac{u_{i,j} - u_{i,j-1}}{h} & \text{if} \quad j = N. \end{cases} \tag{2.15}$$

Besides in practice there are often not exact values of the functions $\alpha(x)$, $\beta(x)$, but approximate values disturbed with some errors.

As long as using the formulae (2.15) could lead to instability of the method, it is reasonable to use another variants of difference approximations presented in the book [15]:

$$\left.\frac{\partial u}{\partial x}\right|_{\substack{t=t_i \\ x=x_j}} = \begin{cases} \dfrac{-21u_{i,j} + 13u_{i,j+1} + 17u_{i,j+2} - 9u_{i,j+3}}{20h}, & \text{if} \quad j = 0, \\ \dfrac{-11u_{i,j-1} + 3u_{i,j} + 7u_{i,j+1} + u_{i,j+2}}{10h}, & \text{if} \quad j = 1, \\ \dfrac{-2u_{i,j-2} - u_{i,j-1} + u_{i,j+1} + 2u_{i,j+2}}{10h}, & \text{if} \quad 2 \leqslant j \leqslant N-2, \\ \dfrac{-11u_{i,j+1} + 3u_{i,j} + 7u_{i,j-1} + u_{i,j-2}}{20h}, & \text{if} \quad j = N-1, \\ \dfrac{-21u_{i,j} + 13u_{u,j-1} + 17u_{i,j-2} - 9u_{i,j-3}}{20h}, & \text{if} \quad j = N. \end{cases} \tag{2.16}$$

Finally it should be mentioned that although the condition (2.4) significantly simplifies the method it is not necessary for using it. Actually the condition (2.4) could be excluded and the steps h and τ could be defined independently from each other. But in this case values $x_j \pm a t_i$ in the general case will not coincide with nodes of a uniform net, and in order to compute integrals on the right side of the equation (2.5) we need to use local splines: this approach will increase computational complexity of the method and at the same time decrease its accuracy.

R e m a r k 2.4 *Implementation of the computational scheme described above requires reduction of a number of nodes by two per each successive layer with respect the t variable. In order to reconvert function values* $u(t_i, x_j)$ *for* $i = \overline{1, M}$, $j = \overline{0, i-1}$, $j = \overline{N-i+1, N}$ *at the missed nodes we can construct a special difference scheme be analogy with the scheme proposed in the paper* [16] *for analysis of potential fields.*

3. Solving model examples

E x a m p l e 3.1 *Let it be required to recover values of function $u(t,x)$ which is the solution of Cauchy problem*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \left(x^4 + 2t\right) u + 4tu\ln(u), \qquad 0 \leqslant t \leqslant 1, \tag{3.1}$$

$$u(0,x) = 1, \tag{3.2}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -x^2. \tag{3.3}$$

R e m a r k 3.1 *The exact solution of the problem is given by the following formula*

$$u(t,x) = e^{-tx^2}. \tag{3.4}$$

It is supposed that the functions $\alpha(x) = u(0,x)$ and $\beta(x) = \left.\frac{\partial u}{\partial t}\right|_{t=0}$ are defined as the finite sets $\{\alpha(x_j)\}_{j=\overline{0,N}}$ and $\{\beta(x_j)\}_{j=\overline{0,N}}$, where $x_j = -A + jh$. In this example we define $A = 2$, $N = 400$, $h = 0.01$.

The stated problem is solved with the proposed method. We set the following values of parameters of the method.

$$L = 100, \quad \theta = 0.1.$$

Due to the condition (2.4) we have $\tau = 0.01$. Therefore $\widetilde{T} = 1$ and consequently $T = \widetilde{T}$. The values $\overline{u}_{i,j}(0)$ for all $i = \overline{1,M}$, $j = \overline{i, N-i}$ are set to 1.

The results of computational experiments are presented in the table 3.1.

Table 3.1. Approximate solution of the equation (3.1)

t_i	ε_i	t_i	ε_i	t_i	ε_i	t_i	ε_i
0.05	0.003421	0.30	0.012163	0.55	0.016164	0.80	0.018916
0.10	0.005983	0.35	0.013174	0.60	0.016746	0.85	0.019499
0.15	0.007992	0.40	0.014058	0.65	0.017296	0.90	0.020132
0.20	0.009627	0.45	0.014838	0.70	0.017831	0.95	0.020832
0.25	0.010995	0.50	0.015534	0.75	0.018365	1.00	0.021620

The error values for layers $t = t_i$ are listed with step equals to 0.05. The error ε_i is defined as the greatest over all nodes x_j ($j = \overline{i, N-i}$) absolute value of difference between exact and approximate values of $u(t,x)$ at node $u(t,x)$.

R e m a r k 3.2 *Let us find approximate solutions of the following problem:*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\cdot\left(\frac{\partial u}{\partial x}\right)^2 - \frac{x^4 + 2t}{(1 + tx^2)^2}, \quad 0 \leqslant t \leqslant 1, \tag{3.5}$$

$$u(0,x) = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x^2. \tag{3.6}$$

Remark 3.3 *The exact solution of the problem* (3.5)-(3.6) *is defined by the formula*

$$u(t,x) = \ln\left(1 + tx^2\right). \tag{3.7}$$

Suppose that values of the functions $\alpha(x)$, $\beta(x)$ are known on the interval $x \in [-3/2, 3/2]$ on a uniform mesh with the step $h = 0.01$. As far as it can be seen from (3.6) *$a = 1$; then due to* (2.4) *we have $\tau = 0.01$. After setting $M = 100$ we get $\widetilde{T} = M\tau = 1 = T$. For Euler method we set the following values of parameters: $\theta = 0.1$, $L = 100$.*

The results of computational experiments are presented in the table 3.2.

Table 3.2. Approximate solution of the equation (3.5)

t_i	ε_i	t_i	ε_i	t_i	ε_i	t_i	ε_i
0.050000	0.000983	0.300000	0.002657	0.550000	0.003229	0.800000	0.004348
0.100000	0.001630	0.350000	0.002768	0.600000	0.003390	0.850000	0.004675
0.150000	0.002051	0.400000	0.002870	0.650000	0.003580	0.900000	0.005040
0.200000	0.002329	0.450000	0.002975	0.700000	0.003802	0.950000	0.005443
0.250000	0.002519	0.500000	0.003092	0.750000	0.004057	1.000000	0.005883

4. Conclusion

In the paper we propose a numerical method for approximate solution of nonlinear hyperbolic differential equations. It is based on continuous operator method for solution of nonlinear operator equations in Banach spaces. The proof of its convergence offered numerical algorithms is based on the Liapunov theory of stability of solutions of ODE. The most striking practical advantage can be revealed for nonlinear integral equations. When solving such equations within more traditional Newton-Kantorovich approach, existence of the inverse of the Frechet derivative for each iterative step is required. It also requires solving a system of linear equations at each step, which can be computationally prohibitive. In contrast, our approach allows us to calculate the solution without such a restriction. The evolutionary nature of the method makes the broad set of methods available for solving ODE — including recurrent neural network techniques — be applied to solving linear and nonlinear hypersingular integral equations. Solving model examples illustrates efficiency of the proposed method.

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Submitted 04.03.2020