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Динамические бифуркационные задачи со спектром Э.Шмидта в линеаризации в условиях групповой симметрии

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Аннотация. Результаты работ [1], [2] для стационарных задач теории ветвления со спектром Э.Шмидта в линеаризации трансформируются на динамические бифуркационные задачи на спектре Э.Шмидта. На основе общей теоремы о наследовании групповой симметрии нелинейной задачи соответствующими уравнениями разветвления в корневых подпространствах (УРК), движущимися по траектории точки ветвления доказана теорема о неявных операторах в условиях групповой симметрии и теорема о редукции УРК по числу уравнений в случае вариационных УРК. Использованы терминология и обозначения [3]– [6].

Ключевые слова: динамические бифуркационные задачи; бифуркация Пуанкаре-Андронова-Хопфа; спектр Шмидта; групповая симметрия; G-инвариантная теорема о неявных операторах; бифуркация; устойчивость; спектр Э.Шмидта; уравнение разветвления в корневых подпространствах вариационного типа

1. Introduction.

In cycle of works at the beginning of XX century on linear and nonlinear integral equations E.Schmidt had introduced eigenvalues λ_k of an operator acting in a Hilbert space $B: H \to H$, taking into account their multiplicities and eigenelements $\{u_k\}_1^\infty, \{v_k\}_1^\infty$ satisfying the relations $Bu_k = \lambda_k v_k$, $B^* v_k = \lambda_k u_k$, that allows to extend Hilbert-Schmidt theory on nonsymmetric completely continuous operators in abstract separable Hilbert spaces. Later such eigenvalues get the name s-numbers (we have introduced in our previous articles the notion "Schmidt spectrum"). Since this article is the direct prolongation of the work [1], where stationary bifurcation problems on E.Schmidt spectrum were considered, here as far as possible auxiliary material connected with E.Schmidt spectrum and its applications contained in [1] and more earlier articles will be reduced. Indicated there possible applications to electromagnetic oscillations theory state the problem on bifurcation and stability of bifurcating solutions in dynamic bifurcational problems with E.Schmidt spectrum in the linearization, in particular under group symmetry conditions. The aim of this article is the transformation of the results [1] on dynamic bifurcation problems. These are the group symmetry inheritance theorem by the relevant branching equations (BEq) and branching equations in the root-subspaces (BEqRs) with corollaries:

1. G-invariant implicit operators theorem [7];

2. theorem on reduction by the the order of variational and variational type BEqs and BEqRs for the cases of non-invariant zero-subspaces of operators [2], [8], [9].

The obtained results are supposed to apply to some problems of electromagnetic oscillations theory.

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2. Poincarè-Andronov-Hopf bifurcation on E. Schmidt spectrum.

In real Banach spaces E_1 and E_2, $E_1 \subset E_2 \subset H$ (H is a Hilbert space) the system of implicitly given differential equations non-resolved to derivatives is considered

$$\begin{gathered} F_1(p_1,p_2,x,y,\varepsilon)=0,\ F_2(p_1,p_2,x,y,\varepsilon)=0,\ F_k(0,0,x_0,y_0,\varepsilon)\equiv 0,\ k=1,2,\ p_1=\tfrac{dx}{dt},\ p_2=\tfrac{dy}{dt}, \\ F'_{1p_1}(0,0,x_0,y_0,\varepsilon)=D_0+D_0(\varepsilon),\ F'_{2p_2}(0,0,x_0,y_0,\varepsilon)=-A_0-A_0(\varepsilon), \\ F'_{2p_1}(0,0,x_0,y_0,\varepsilon)=-A_0^*-A_0^*(\varepsilon),\ F'_{2p_2}(0,0,x_0,y_0,\varepsilon)=D_0^*+D_0^*(\varepsilon), \\ F'_{1x}(0,0,x_0,y_0,\varepsilon)=-B_0+B_0(\varepsilon),\ F'_{1y}(0,0,x_0,y_0,\varepsilon)=C_0-C_0(\varepsilon), \\ F'_{2x}(0,0,x_0,y_0,\varepsilon)=C_0^*-C_0^*(\varepsilon),\ F'_{2y}(0,0,x_0,y_0,\varepsilon)=-B_0^*+B_0^*(\varepsilon) \end{gathered} \tag{2.1}$$

In general case when the operators $A_0, A_0(\varepsilon), ..., D_0, D_0(\varepsilon)$, and adjoint to them can be unbounded it is supposed that $D_{A_0} \subset D_{A_0(\varepsilon)}, \overline{D}_{A_0} = E_1, ..., D_{D_0} \subset D_{D_0(\varepsilon)}, \overline{D}_{D_0} = E_1$ the system (2.1) allows the following linearization

$$\begin{gathered} \begin{pmatrix} -A_0^* & D_0^* \\ D_0 & -A_0 \end{pmatrix}\begin{pmatrix} x'_t \\ y'_t \end{pmatrix} = \left[\begin{pmatrix} -C_0^* & B_0^* \\ B_0 & -C_0 \end{pmatrix} - \begin{pmatrix} -C_0^*(\varepsilon) & B_0^*(\varepsilon) \\ B_0(\varepsilon) & -C_0(\varepsilon) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} - \right. \\ \left. -\begin{pmatrix} -A_0^*(\varepsilon) & D_0^*(\varepsilon) \\ D_0(\varepsilon) & -A_0(\varepsilon) \end{pmatrix}\begin{pmatrix} x'_t \\ y'_t \end{pmatrix} - \begin{pmatrix} R_2(x-x_0,y-y_0,x'_t,y'_t,\varepsilon) \\ R_1(x-x_0,y-y_0,x'_t,y'_t,\varepsilon) \end{pmatrix}\right] \end{gathered} \tag{2.2}$$

which is convenient to present in the matrix form

$$\mathbf{A}_0(X'_t,Y'_t)^T = (\mathbf{B}_0-\mathbf{B}_0(\varepsilon))(X,Y)^T - \mathbf{A}_0(\varepsilon)(X'_t,Y'_t)^T + \mathbf{R}(x_0,y_0,X,Y,X'_t,Y'_t,\varepsilon) \tag{2.3}$$

The index zero at the operator and everywhere below means the relation to the point x_0, y_0, i.e. $\mathbf{A}_0 = \mathbf{A}(x_0,y_0)$. The vectorial nonlinear operator $\mathbf{R}$ is supposed to be sufficiently smooth on $\mathcal{X} = (X,Y)^T = (x-x_0, y-y_0)^T$ and $\mathcal{X}'_t = (X'_t,Y'_t)^T$ and $\mathbf{R}(x_0,y_0,0,0,\varepsilon) \equiv 0, \mathbf{R}_{\mathcal{X}}(x_0,y_0,0,\mathcal{X}_t,\varepsilon) = 0, \mathbf{R}_{\mathcal{X}'}(x_0,y_0,\mathcal{X},0,\varepsilon) \equiv 0$. Keeping in mind the dense embedding $E_1 \subset E_2 \subset H$ the operators $\mathbf{A}_0, \mathbf{B}_0, \mathbf{A}_0(\varepsilon), \mathbf{B}_0(\varepsilon)$ can be regarded as acting in the direct sum H^2 of two Hilbert space H.

Further it is considered the sufficiently general case when the $\mathbf{A}_0$-spectrum $\sigma_{\mathbf{A}_0}(\mathbf{B}_0)$ of the Fredholm operator $\mathbf{B}_0$ is decomposed into two parts: $\sigma^-_{\mathbf{A}_0}(\mathbf{B}_0)$ lying strictly in the left half-plane and $\sigma^0_{\mathbf{A}_0}(\mathbf{B}_0)$ consisting of the eigenvalues $\pm i\alpha$ of the multiplicity n. More general case implies only technical difficulties.

Let there exist [10], [11] elements $U_{1k} = U_{1k}^{(1)} = \begin{pmatrix} u_{1k} \\ \widetilde{u}_{1k} \end{pmatrix}, U_{2k} = U_{2k}^{(1)} = \begin{pmatrix} u_{2k} \\ \widetilde{u}_{2k} \end{pmatrix}$ and $\mathcal{V}_{1k} = V_{1k}^{(1)} = \begin{pmatrix} v_{1k} \\ \widetilde{v}_{1k} \end{pmatrix}, V_{2k} = V_{2k}^{(1)} = \begin{pmatrix} v_{2k} \\ \widetilde{v}_{2k} \end{pmatrix}$ belonging to direct sum $H \dot{+} H = H^2$, such that

$$\begin{gathered} \mathcal{B}(\alpha)\Phi_k^{(1)} \equiv \begin{pmatrix} \mathbf{B}_0 & \alpha\mathbf{A}_0 \\ -\alpha\mathbf{A}_0 & \mathbf{B}_0 \end{pmatrix}\begin{pmatrix} U_{1k}^{(1)} \\ U_{2k}^{(1)} \end{pmatrix} = 0, \mathcal{B}^*(\alpha)\Psi_k^{(1)} \equiv \begin{pmatrix} \mathbf{B}_0^* & -\alpha\mathbf{A}_0^* \\ \alpha\mathbf{A}_0^* & \mathbf{B}_0^* \end{pmatrix}\begin{pmatrix} V_{1k}^{(1)} \\ V_{2k}^{(1)} \end{pmatrix} = 0, \\ k=1,...,n \end{gathered} \tag{2.4}$$

It means that the zero-subspaces $\mathcal{N}(\mathcal{B}(\alpha))$ and $\mathcal{N}(\mathcal{B}^*(\alpha))$ of the operators $\mathcal{B}(\alpha)$ and $\mathcal{B}^*(\alpha)$ have the forms $\mathcal{N}(\mathcal{B}(\alpha)) = \operatorname{span}\{\Phi_{1k}^{(1)} = \begin{pmatrix} U_{1k} \\ U_{2k} \end{pmatrix}, \Phi_{2k}^{(1)} = \begin{pmatrix} U_{2k} \\ -U_{1k} \end{pmatrix}, k = 1,...,n\}$, $\mathcal{N}(\mathcal{B}^*(\alpha)) = \operatorname{span}\{\Psi_{1k}^{(1)} = \begin{pmatrix} -V_{2k} \\ V_{2k} \end{pmatrix}, \Psi_{2k}^{(1)} = \begin{pmatrix} V_{1k} \\ V_{2k} \end{pmatrix}, k = 1,...,n\}$, and in coordinate representation respectively

$$\begin{array}{rr} -C_0^*u_{1k}+B_0^*\widetilde{u}_{1k}-\alpha A_0^*u_{2k}+\alpha D_0^*\widetilde{u}_{2k}=0 & -C_0^*v_{1k}+B_0^*\widetilde{v}_{1k}+\alpha A_0^*v_{2k}-\alpha D_0^*\widetilde{v}_{2k}=0 \\ B_0u_{1k}-C_0\widetilde{u}_{1k}+\alpha D_0u_{2k}-\alpha A_0\widetilde{u}_{2k}=0 & B_0v_{1k}-C_0^*\widetilde{v}_{1k}-\alpha D_0v_{2k}+\alpha A_0^*\widetilde{v}_{2k}=0 \\ \alpha A_0^*u_{1k}-\alpha D_0^*\widetilde{u}_{1k}-C_0^*u_{2k}+B_0^*\widetilde{u}_{2k}=0 & -\alpha A_0v_{1k}+\alpha D_0^*\widetilde{v}_{1k}-C_0v_{2k}+B_0^*\widetilde{v}_{2k}=0 \\ -\alpha D_0u_{1k}+\alpha A_0\widetilde{u}_{1k}+B_0u_{2k}-C_0\widetilde{u}_{2k}=0 & \alpha D_0v_{1k}-\alpha A_0^*\widetilde{v}_{1k}+B_0v_{2k}-C_0^*\widetilde{v}_{2k}=0 \end{array} \qquad (2.5)$$

that can be regarded as the systems in the direct sum $\mathcal{H}$ of four Hilbert spaces.

Carrying out the complexification of the equation (2.1), consider it in the spaces $\mathcal{E}_k = E_k \dot{+} iE_k, k=1,2$ and suppose that the nonlinear operator $\mathbf{R}$ admits a sufficiently smooth extension on these spaces. Then the elements $U_k = U_{1k}+iU_{2k} = \begin{pmatrix} u_{1k}+iu_{2k} \\ \widetilde{u}_{1k}+i\widetilde{u}_{2k}\end{pmatrix}, \overline{U}_k$ and $V_k = V_{1k}+iV_{2k} = \begin{pmatrix} v_{1k}+iv_{2k} \\ \widetilde{v}_{1k}+i\widetilde{v}_{2k}\end{pmatrix}, \overline{V}_k$ are the eigenelements of the following eigenvalue problems

$$\mathbf{B}_0U_k = i\alpha\mathbf{A}_0U_k,\ \mathbf{B}_0\overline{U}_k = -i\alpha\mathbf{A}_0\overline{U}_k,\ \mathbf{B}_0^*V_k = -i\alpha\mathbf{A}_0V_k,\ \mathbf{B}_0^*\overline{V}_k = i\alpha\mathbf{A}_0\overline{V}_k,\ k=1,...,n \quad (2.6)$$

It can be easily verified: the substitution U_k, V_k or $\overline{U}_k, \overline{V}_k$ into the relations (2.6) after the separation of real and imaginary parts leads to the systems (2.5).

The problem of the finding of $\frac{2\pi}{\alpha+\mu}$-periodical solutions to (2.3) is setting,where $\mu=\mu(\varepsilon)\to 0, \varepsilon\to 0$ is the unknown addition to the frequency of oscillations. The Poincarè substitution $t=\frac{\tau}{\alpha+\mu}, \mathcal{X}(t) = (X(t),Y(t))^T = \mathcal{Y}(\tau)$ reduces this problem to the determination of 2π-periodic solutions of the following equation with two small parameters μ and ε

$$\begin{gathered} \mathfrak{B}_0\mathcal{Y} = \mu\mathfrak{A}_0\mathcal{Y} + (\alpha+\mu)\mathfrak{A}_0(\varepsilon)\mathcal{Y} + \mathfrak{B}_0(\varepsilon)\mathcal{Y} + \mathbf{R}(x_0,y_0,(\alpha+\mu)\tfrac{d\mathcal{Y}}{dt},\mathcal{Y},\varepsilon) \equiv \\ \equiv \mu\mathfrak{A}_0\mathcal{Y} + \mathcal{R}(x_0,y_0,\tfrac{d\mathcal{Y}}{dt},\mathcal{Y},\mu,\varepsilon), \\ (\mathfrak{B}_0\mathcal{Y})(\tau) \equiv \mathfrak{B}_0\mathcal{Y} = \mathbf{B}_0\mathcal{Y}(\tau) - \alpha\mathbf{A}_0\tfrac{d\mathcal{Y}}{d\tau},\ (\mathfrak{A}_0\mathcal{Y})(\tau) = \mathfrak{A}_0\mathcal{Y} = \mathbf{A}_0\tfrac{d\mathcal{Y}}{d\tau}, \\ \mathfrak{A}_0(\varepsilon)\mathcal{Y} = (\mathfrak{A}_0(\varepsilon)\mathcal{Y})(\tau) = \mathbf{A}_0(\varepsilon)\tfrac{d\mathcal{Y}}{d\tau} \end{gathered} \qquad (2.7)$$

The supposed Fredholmian operator $\mathfrak{B}_0\mathcal{Y}$ and operators in (2.7) are mapping the space $\mathbb{Y}$ of 2π-periodic continuously differentiable functions τ with values in $\mathcal{E}_1^2 \subset \widetilde{\mathcal{H}}$ into the space $\mathbb{Z}$ of 2π-periodic continuously differentiable functions τ with values in $\mathcal{E}_2^2 \subset \widetilde{\mathcal{H}}$ at the usage of special form functionals $\langle\langle \mathcal{Y},\mathcal{F}\rangle\rangle = \int\limits_0^{2\pi}\langle\mathcal{Y}(\tau),\mathcal{F}(\tau)\rangle$, $\mathcal{Y}\in\mathbb{Y}\subset\widetilde{\mathcal{H}}, \mathcal{F}\in\mathbb{Y}^*\subset\widetilde{\mathcal{H}}$ or $\mathcal{Y}\in\mathbb{Z}\subset\widetilde{\mathcal{H}}, \mathcal{F}\in\mathbb{Z}^*\subset\widetilde{\mathcal{H}}$. Zero-subspaces of the operators $\mathfrak{B}_0$ and $\mathfrak{B}_0^*$ are $2n$-dimensional

$$\begin{gathered} \mathcal{N}(\mathfrak{B}_0) = \mathrm{span}\{\varphi_k^{(1)} = \varphi_k^{(1)}(x_0,y_0,\tau)\} = U_k(x_0,y_0)e^{i\tau}, \overline{\varphi}_k^{(1)}\}_{k=1}^n \\ \mathcal{N}(\mathfrak{B}_0^*) = \mathrm{span}\{\psi^{(1)} = \psi_k^{(1)}(x_0,y_0,\tau)\} = V_k(x_0,y_0)e^{i\tau}, \overline{\psi}_k^{(1)}\}_{k=1}^n \end{gathered}$$

with $\mathbf{A}_0$- and $\mathbf{A}_0^*$-Jordan chains ($\mathfrak{A}_0$- and $\mathfrak{A}_0^*$- Jordan chains) $\varphi_k^{(s)} = U_k^{(s)}(x_0,y_0)e^{i\tau}, \psi_k^{(s)} = V_k^{(s)}(x_0,y_0)e^{i\tau}, s=\overline{1,p_k}$ of the length $p_k, k=\overline{1,n}$. Here $\mathcal{H}$ is a Hilbert space with elements of $\widetilde{\mathcal{H}}$ containing the factors $e^{im\tau}, m$- are integers.

Definition 2.1. [12], [13]. *The elements* $U_k^{(s)}(x_0,y_0) = (u_{1k}^{(s)}+iu_{2k}^{(s)}, \widetilde{u}_{1k}^{(s)}+i\widetilde{u}_{2k}^{(s)}), s=\overline{1,p_k}, k=\overline{1,n}$ *form the complete canonical generalized Jordan set (GJS≡* $\mathbf{A}_0(\varepsilon)$ *-JS), if*

$$(\mathbf{B}_0 - i\alpha\mathbf{A}_0)U_k^{(s)} = \sum_{j=1}^{s-1}\mathbf{A}_jU_k^{(s-j)}, \mathbf{A}_0(\varepsilon) = \mathbf{A}_1\varepsilon+\mathbf{A}_2\varepsilon^2+..., \left\langle U_k^{(s)},\Gamma_l^{(1)}\right\rangle_{\mathcal{H}} = 0, s=2,...,p_k;$$

$$D_p = \det\left[\sum_{j=1}^{s-1}\left\langle \mathbf{A}_jU_k^{(p_k+1-j)}, V_l^{(1)}\right\rangle_{\mathcal{H}}\right] \neq 0, V_l^{(1)} = V_{1l}^{(1)} + iV_{2l}^{(1)} = V_l = (v_{1l}+iv_{2l}, \widetilde{v}_{1k}+i\widetilde{v}_{2l})^T$$

This JS is bicanonical, if GJS of elements $\{V_l^{(1)}\}_1^n$ *for conjugate operator* $(\mathbf{B}_0^*+i\alpha\mathbf{A}_0^*)-\mathbf{A}_0^*(\varepsilon)$ *is also canonical, and three-canonical if in addition*

$$\left\langle U_i^{(j)},\Gamma_k^{(l)}\right\rangle_{\mathcal{H}}=\delta_{ik}\delta_{jl},\ \Gamma_k^{(l)}=\sum_{s=1}^{p_k+1-l}\mathbf{A}_s^*V_k^{(p_k+2-l-s)},$$
$$\left\langle Z_i^{(j)},V_k^{(l)}\right\rangle_{\mathcal{H}}=\delta_{ik}\delta_{jl},\ Z_k^{(j)}=\sum_{s=1}^{p_k+1-i}\mathbf{A}_s^*U_i^{(p_k+2-j-s)}.$$

Later turn out to be convenient the following designations: $U=U(x_0,y_0)=(U_1^{(1)},...,U_1^{(p_1)},...,U_n^{(1)},...,U_n^{(p_n)}),U_i^{(j)}=U_i^{(j)}(x_0,y_0)$, the vectors $\Gamma=\Gamma(x_0,y_0),V=V(x_0,y_0)$ and $Z=Z(x_0,y_0)$ are defined analogously, $K=\sum_{k=1}^{n}p_k$ is the root-number.

R e m a r k. All these notions and designations are naturally transferring on Jordan sets of the zero-subspaces $\mathcal{N}(\mathfrak{B})$ and $\mathcal{N}(\mathfrak{B}^*)$, among them the Jordan chains biorthogonality conditions of the type (2.7).

However in the Poincarè substitution the small parameter μ is depended on ε, i.e. $\mu=\mu(\varepsilon)$ and together with the solution $\mathcal{Y}$ of the equation (2.7) is also be subjected to the determination. Moreover later $\mathbf{A}_0$-Jordan structure of the operator $\mathfrak{B}_0$ will be used because of in the general case $\mu(\varepsilon)$ turns out to be analytic on fractional degrees of ε.

L e m m a 2.1. [5], [6], [11], [12]. *The Fredholmian operator-function* $\mathfrak{B}_0-\mu\mathfrak{A}_0$ *as depending linearly on small parameter* μ *always has complete three-canonical GJS satisfying the biorthogonality conditions*

$$\begin{gathered}\langle\langle\varphi_j^{(k)},\gamma_s^{(l)}\rangle\rangle_{\widetilde{\mathcal{H}}}=\delta_{js}\delta_{kl},\quad \langle\langle z_j^{(k)},\psi_s^{(l)}\rangle\rangle_{\widetilde{\mathcal{H}}}=\delta_{js}\delta_{kl},\quad k(l)=\overline{1,p_j(p_s)},\\ \gamma_s^{(l)}=\mathbf{A}_0\psi_s^{(p_s+1-l)},z_j^{(k)}=\mathbf{A}_0\varphi_j^{(p_j+1-k)},\quad j(s)=\overline{1,n}.\end{gathered}\tag{2.8}$$

The relations (2.8) *allow to determine the projectors*

$$\begin{gathered}\mathbf{P}=\mathbf{P}(x_0,y_0)=\sum_{j=1}^{n}\sum_{k=1}^{p_j}\langle\langle\cdot,\gamma_j^{(k)}\rangle\rangle\varphi_j^{(k)}=\langle\langle\cdot,\gamma\rangle\rangle\varphi,\quad \overline{\mathbf{P}}=\overline{\mathbf{P}}(x_0,y_0)=\langle\langle\cdot,\overline{\gamma}\rangle\rangle\overline{\varphi},\\ \mathbf{Q}=\mathbf{Q}(x_0,y_0)=\sum_{j=1}^{n}\sum_{k=1}^{p_j}\langle\langle\cdot,\psi_j^{(k)}\rangle\rangle z_j^{(k)}=\langle\langle\cdot,\psi\rangle\rangle z,\quad \overline{\mathbf{Q}}=\overline{\mathbf{Q}}(x_0,y_0)=\langle\langle\cdot,\overline{\psi}\rangle\rangle\overline{z},\\ \mathbb{P}(x_0,y_0)=\mathbf{P}(x_0,y_0)+\overline{\mathbf{P}}(x_0,y_0),\quad \mathbb{Q}(x_0,y_0)=\mathbf{Q}(x_0,y_0)+\overline{\mathbf{Q}}(x_0,y_0)\end{gathered}\tag{2.9}$$

generating the decompositions of Hilbert space $\widetilde{\mathcal{H}}$ *(Banach spaces* $\mathbb{Y}$ *and* $\mathbb{Z}$*) into the direct sums*

$$\widetilde{\mathcal{H}}=\widetilde{\mathcal{H}}^{2K}(x_0,y_0)\dot{+}\widetilde{\mathcal{H}}^{\infty-2K}(x_0,y_0),\quad \widetilde{\mathcal{H}}=\widetilde{\mathcal{H}}_{2K}(x_0,y_0)\dot{+}\widetilde{\mathcal{H}}_{\infty-2K}(x_0,y_0)\tag{2.10}$$

Operators $\mathfrak{B}_0$ *and* $\mathbf{A}_0$ $(\mathfrak{A}_0)$ *are intertwining by the projectors* $\mathbf{P}(x_0,y_0)$ *and* $\mathbf{Q}(x_0,y_0)$, $\overline{\mathbf{P}}(x_0,y_0)$ *and* $\overline{\mathbf{Q}}(x_0,y_0)$: $\mathfrak{B}_0\mathbf{P}(x_0,y_0)u=\mathbf{Q}(x_0,y_0)\mathfrak{B}_0u$ *on* $D_{\mathfrak{B}_0},\mathfrak{B}_0\varphi=\mathfrak{V}_0z,\mathfrak{B}_0^*\psi=\mathfrak{V}_0\gamma,$ $\mathfrak{V}_0$ *is is cell-diagonal matrix* $\mathfrak{V}_0=diag(B_1,\ldots,B_n)$, *where* B_i – $(p_i\times p_i)$ *- matrix with units on subsidiary subdiagonal and zeros on other places;* $\mathfrak{A}_0\mathbf{P}(x_0,y_0)u=\mathbf{Q}(x_0,y_0)\mathfrak{A}_0u$ *on* $D_{\mathfrak{A}_0},\mathbf{A}_0\varphi=\mathfrak{A}_0z,\mathbf{A}_0\psi=\overline{\mathfrak{V}}_1\gamma$, *where* $\overline{\mathfrak{V}}_1$ *is cell-diagonal matrix* $\overline{\mathfrak{V}}_1=diag(B^1,\ldots,B^n)$, B^i *is* $(p_i\times p_i)$ *matrix with units on subsidiary diagonal and zeros on other places; Operators* $\mathfrak{B}_0$ *and* $\mathfrak{A}_0$ $(\mathbf{A}_0)$ *are acting in invariant pairs of subspaces* $\widetilde{\mathcal{H}}^{\infty-2K}(x_0,y_0)$ *and* $\widetilde{\mathcal{H}}_{\infty-2K}(x_0,y_0)$, $\widetilde{\mathcal{H}}^{2K}(x_0,y_0)$ *and* $\widetilde{\mathcal{H}}_{2K}(x_0,y_0)$ *and* $\mathfrak{B}_0:D_{\mathfrak{B}_0}\bigcap\widetilde{\mathcal{H}}^{\infty-2K}(x_0,y_0)\to\widetilde{\mathcal{H}}_{\infty-2K}(x_0,y_0),\mathbf{A}_0:D_{\mathbf{A}_0}\bigcap\widetilde{\mathcal{H}}^{2K}(x_0,y_0)\to\widetilde{\mathcal{H}}_{2K}(x_0,y_0)$ *are isomorphisms.*

T h e o r e m 2.1. *In conditions of Lemma 2.1 the problem on periodic solutions the equation* (2.3) *(*2π*-periodic solutions to the* (2.7)*) in a neighborhood of the bifurcation point* $(x_0,y_0;0)$ *is equivalent to the finding of small solutions to finite-dimensional A.M. Lyapounov BEqR* (2.11) *or E. Schmidt* (2.12).

Proof. In accordance with the expansion (2.10) setting $\mathcal{Y}=\mathcal{U}+\mathcal{V},\mathcal{V}=\mathcal{V}(x_0,y_0,\xi,\overline{\xi})=\sum(\xi_{jk}\varphi_j^{(k)}(x_0,y_0)+\overline{\xi}_{jk}\overline{\varphi}_j^{(k)}(x_0,y_0))=\xi\cdot\varphi+\overline{\xi}\cdot\overline{\varphi}\in\widetilde{\mathcal{H}}^{2K}(x_0,y_0),\mathcal{U}=\mathcal{U}(x_0,y_0)\in\widetilde{\mathcal{H}}^{\infty-2K}(x_0,y_0)$ write the equation (2.7) in projections on the root-subspaces and their direct supplements in the point (x_0,y_0)

$$\begin{aligned}&(I-\mathbb{Q}(x_0,y_0)\mathfrak{B}_0(x_0,y_0)\mathcal{U}(x_0,y_0)=\\&=(I-\mathbb{Q}(x_0,y_0))\{\mu\mathfrak{A}_0(\mathcal{U}(x_0,y_0)+\mathcal{V}(x_0,y_0)+\mathcal{R}(x_0,y_0,\tfrac{d[...]}{d\tau},[...],\mu,\varepsilon))\},\\&\mathbb{Q}(x_0,y_0)\mathfrak{B}_0\mathcal{V}(x_0,y_0)=\mathbb{Q}(x_0,y_0)\{\mu\mathfrak{A}_0(\mathcal{U}(x_0,y_0)+\mathcal{V}(x_0,y_0)+\mathcal{R}(x_0,y_0,\tfrac{d[...]}{d\tau},[...],\mu,\varepsilon))\}\end{aligned}$$

and resolving the first of them according implicit operators theorem [3] and Lemma 2.1 with respect to $\mathcal{U}(x_0,y_0)$ we obtain $\mathcal{U}(x_0,y_0)=\mathcal{U}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)$. Its substitution in the second equation gives A.Lyapounov BEqR

$$\begin{aligned}&\mathbf{f}(x_0,y_0;\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=\mathbf{Q}(x_0,y_0)\{\mu\mathfrak{A}_0(\mathcal{U}+\mathcal{V})+\mathcal{R}(x_0,y_0,\tfrac{d}{d\tau}[...],[...],\mu,\varepsilon)=\\&\quad=(\mathfrak{V}_0-i\mu\mathfrak{V}_1)\xi-\langle\langle\mathcal{R}(x_0,y_0,\tfrac{d}{d\tau}[...],[...],\mu,\varepsilon),\psi(x_0,y_0)\rangle\rangle=0,\\&\overline{\mathbf{f}}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=0,\quad[...]=\mathcal{U}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)+\mathcal{V}(x_0,y_0,\xi,\overline{\xi}).\end{aligned}\tag{2.11}$$

More detaily consider the construction of E.Schmidt BEqR. Writing the equation (2.7) in the form of the equivalent system

$$\begin{aligned}&\widetilde{\mathfrak{B}}_0\,\mathcal{Y}=\mu\mathfrak{A}_0\mathcal{Y}+\mathcal{R}(x_0,y_0,\tfrac{d\mathcal{Y}}{d\tau},\mathcal{Y},\mu,\varepsilon)+\sum_{i=1}^{n}(\xi_{i1}z_i^{(1)}+\overline{\xi}_{i1}\overline{z}_i^{(1)}),\\&\xi_{j\sigma}=\langle\langle\mathcal{Y},\gamma_s^{(\sigma)}\rangle\rangle,\overline{\xi}_{s\sigma}=\langle\langle\mathcal{Y},\overline{\gamma}_s^{(\sigma)}\rangle\rangle,\end{aligned}\tag{2.12}$$

where $\widetilde{\mathfrak{B}}_0=\mathfrak{B}_0(x_0,y_0)+\sum_{i=1}^{n}[\langle\langle\cdot,\gamma_i^{(1)}\rangle\rangle z_i^{(1)}+\langle\langle\cdot,\overline{\gamma}_i^{(1)}\rangle\rangle\overline{z}_i^{(1)}]$ is E.Schmidt regularizator [3], $\widetilde{\mathfrak{B}}_0^{-1}=\Gamma(x_0,y_0)=\Gamma_0$, its solution find in the form $\mathcal{Y}=w+\mathcal{V}(x_0,y_0,\xi,\overline{\xi})=w+\xi\cdot\varphi+\overline{\xi}\cdot\overline{\varphi}$. This gives $w=-(I-\mu\Gamma_0\mathfrak{A}_0)^{-1}\Gamma_0\mathfrak{A}_0(\xi\cdot\varphi+\overline{\xi}\cdot\overline{\varphi})+\Gamma_0(I-\mu\mathfrak{A}_0)\Gamma_0)^{-1}\mathcal{R}(x_0,y_0,(\alpha+\mu)\frac{d}{d\tau}[w(x_0,y_0)+\mathcal{V}(x_0,y_0,\xi,\overline{\xi})],w(x_0,y_0)+\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)$. Substitution of $\mathcal{Y}=w+\mathcal{V}$ into the second equations (2.12) taking into account the relations $\Gamma_0^*\gamma_j^{(1)}(x_0,y_0)=\psi_j^{(1)}(x_0,y_0),\Gamma_0^*\gamma_j^{(s)}(x_0,y_0)=\psi_j^{(p_j+2-\sigma)}(x_0,y_0),s\geq 2$, leads to E.Schmidt BEqR in the basis $\{\varphi,\overline{\varphi}\}$

$$\begin{aligned}&t_{s1}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=-\langle\langle w,\gamma_s^{(1)}(x_0,y_0)\rangle\rangle=\\&=-\mu\langle\langle\mathfrak{A}_0(I-\mu\Gamma_0\mathfrak{A}_0)^{-1}(\xi\cdot\varphi+\overline{\xi}\cdot\overline{\varphi}),\psi_s^{(1)}(x_0,y_0)\rangle\rangle-\langle\langle(I-\mu\mathfrak{A}_0\Gamma_0)^{-1}\mathcal{R}(...),\psi_s^{(1)}(x_0,y_0)\rangle\rangle=0,\\&t_{s\sigma}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=-\langle\langle w,\gamma_s^{(\sigma)}(x_0,y_0)\rangle\rangle=\\&=\langle\langle(I-\mu\Gamma_0\mathfrak{A}_0)^{-1}\sum_{i=1}^{n}\sum_{j=2}^{p_i}(\xi_{ij}\varphi_i^{(j)}+\overline{\xi}_{ij}\overline{\varphi}_i^{(j)})-\mu(I-\mu\Gamma_0\mathfrak{A}_0)^{-1}\Gamma_0\mathfrak{A}_0(\xi\cdot\varphi+\overline{\xi}\cdot\overline{\varphi}),\gamma_s^{(\sigma)}(x_0,y_0)\rangle\rangle-\\&-\langle\langle(I-\mu\mathfrak{A}_0\Gamma_0)^{-1}\mathcal{R}(...),\psi_s^{(p_s+2-\sigma)}(x_0,y_0)\rangle\rangle=0.\end{aligned}$$

With regard to the relations $i^{\sigma-1}\varphi_j^{(\sigma)}(x_0,y_0)=(\Gamma_0\mathfrak{A}_0)^{\sigma-1}\varphi_j^{(1)}(x_0,y_0)=i\varphi_j^{\left(\sigma-\left[\frac{\sigma}{p_j}\right]p_j\right)}(x_0,y_0)$ it transforms to the form

$$\begin{aligned}&t_{s1}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=-\tfrac{(i\mu)^{p_s}}{1-(i\mu)^{p_s}}\xi_{s1}-\langle\langle(I-\mu\mathfrak{A}_0\Gamma_0))^{-1}\mathcal{R}(...),\psi_s^{(1)}(x_0,y_0)\rangle\rangle=0,\\&t_{s\sigma}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=\\&=\xi_{s\sigma}-\tfrac{(i\mu)^{\sigma-1}}{1-(i\mu)^{p_s}}\xi_{s1}-\langle\langle(I-\mu\mathfrak{A}_0\Gamma_0)^{-1}\mathcal{R}(...),\psi_s^{(p_s+2-\sigma)}(x_0,y_0)\rangle\rangle=0,\\&\overline{t_{s1}}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=0,\ \ \overline{t_{s\sigma}}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)=0,\\&s=1,...,n,\quad\sigma=1,...,p_s.\end{aligned}\tag{2.13}$$

3. Theorems on group symmetry inheritance of original nonlinear equation by the relevant BEqRs.

At the presence of continuous symmetry admitting by original nonlinear problem (2.1) with operators acting in Banach spaces Lie group $G_l = G_l(a),\ a = (a_1, \ldots, a_l)$ its essential parameters is supposed being l-dimensional differentiable manifold, satisfying following conditions [8],[9],[1],[2],[7]:

$(\mathbf{c_1})$. representation $a \mapsto L_{g(a)}x_0$, acting from a neighborhood of the unit element of $G_l(a)$ into the space E_1 belong to the class C^1, so that $X(x_0, y_0)^T \in E_1 \dot{+} E_1 \subset H^2$ for all infinitesimal operators $X(x,y)^T = \lim\limits_{t\to 0} t^{-1}\left[L_{g(a(t))}(x,y)^T - (x,y)^T\right]$ in tangent to $L_{g(a)}$ manifold $T^l_{g(a)}$;

$(\mathbf{c_2})$. stationary subgroup of the element $(x_0, y_0)^T \in E_1 \dot{+} E_1 \subset H^2$ determines the representation $L(G_s)$ of local Lie group $G_s \subset G_l,\ s < l$, with s-dimensional subalgebra $T^s_{g(a)}$ of infinitesimal operators. This means that for non-stationary bifurcation element $X_k(x_0, y_0)^T, X_k \in T^l_{g(a)}$ form in the zero-subspace of the linearized operator $2\kappa = 2(l-s)$-dimensional subspace and bases in it and in the algebra $T^l_{g(a)}$ can be ordered so that $X_k(x_0, y_0)^T = \xi_k\varphi_k + \overline{\xi}_k\overline{\varphi}_k, 1 \le k \le \kappa, X_j(x_0, y_0)^T = 0$ для $j \ge \kappa + 1$.

$(\mathbf{c_3})$. As earlier the dense embeddings $E_1 \subset E_2 \subset H$ in Hilbert space H with estimates $\|u\|_H \le \alpha_2\|u\|_{E_2} \le \alpha_1\|u\|_{E_1}$ and condition that the mapping $X : E_1 \to H$ is bounded in $L(E_1, H)$ topology.

Everywhere below it is supposed that the system (2.1) allows the group symmetry

$$K_gF_j(p_1, p_2, x, y, \varepsilon) = F_j(L_gp_1, L_gp_2, L_gx, L_gy, \varepsilon), \quad j = 1, 2 \tag{3.1}$$

where $L_g(K_g)$ is the representation of the group G in $E_1(E_2)$ expanded on H. Here the bifurcation point (x_0, y_0) moves along its trajectory (L_gx_0, L_gy_0). When G is a Lie group the conditions $(\mathbf{c_1})$.–$(\mathbf{c_3})$. are supposed to be realized. Similary to [1] the auxiliary constructions are introduced:

$$\begin{aligned}&\mathbf{1^0}.\ K_g[\mathbf{B}_0 \pm i\alpha\mathbf{A}_0] = K_g[\mathbf{B}_0(0,0,x_0,y_0,0) \pm i\alpha\mathbf{A}_0(0,0,x_0,y_0,0)] = \\ &= [\mathbf{B}_0(0,0,L_gx_0,L_gy_0,0) \pm \alpha\mathbf{A}_0(0,0,L_gx_0,L_gy_0)]L_g.\end{aligned} \tag{3.2}$$

In fact, from (3.1) and (2.1), (2.2) it follows

$$\begin{gathered}K_gF'_{j_{p_k}}(0,0,x_0,y_0,\varepsilon) = F'_{j_{p_k}}(0,0,L_gx_0,L_gy_0,\varepsilon),\\ K_gF'_{j_x}(0,0,x_0,y_0,\varepsilon) = F'_{j_{L_gx}}(0,0,L_gx_0,L_gy_0,\varepsilon)L_g(x-x_0),\\ K_gF'_{j_y}(0,0,x_0,y_0,\varepsilon) = F'_{j_{L_gy}}(0,0,L_gx_0,L_gy_0,\varepsilon)L_g(y-y_0),\\ K_gR_j(x_0,y_0,x-x_0,y-y_0,p_1,p_2,\varepsilon) = R_j(L_gx_0,L_gy_0,L_g(x-x_0),L_g(y-y_0),L_gp_1,L_gp_2,\varepsilon)\end{gathered}$$

and the required relations follows from the relevant matrix representation of the operators $\mathbf{B}_0$ and $\mathbf{A}_0$. Analogously the following relations can be required

$$\mathbf{2^0}.\ K_g\mathbf{A}_0(\varepsilon) = K_g\mathbf{A}_0(0,0,x_0,y_0,\varepsilon) = K_g\begin{pmatrix} -F'_{2_{p_1}}(0,0,x_0,y_0,\varepsilon), & F'_{2_{p_2}}(0,0,x_0,y_0,\varepsilon)\\ F'_{1_{p_1}}(0,0,x_0,y_0,\varepsilon), & -F'_{1_{p_2}}(0,0,x_0,y_0,\varepsilon)\end{pmatrix} =$$

$$\begin{pmatrix} -F'_{2_{p_1}}(0,0,L_gx_0,L_gy_0,\varepsilon), & F'_{2_{p_2}}(0,0,L_0x_0,L_0y_0,\varepsilon)\\ F'_{1_{p_1}}(0,0,L_0x_0,L_0y_0,\varepsilon), & -F'_{1_{p_2}}(0,0,L_0x_0,L_)y_0,\varepsilon)\end{pmatrix} = \mathbf{A}_0(0,0,L_gx_0,L_gy_0,\varepsilon)$$

$$\mathbf{3^0}.\ K_g\mathbf{R}(x_0,y_0,x-x_0,y-y_0,p_1,p_2,\varepsilon) = \mathbf{R}(L_gx_0,L_gy_0,L_g(x-x_0),L_g(y-y_0),p_1,p_2,\varepsilon)$$

Consequently

$$\begin{gathered}\varphi_k(L_gx_0,L_gy_0) = L_g\varphi_k(x_0,y_0) = \begin{pmatrix} L_gu^{(1)}_{1k}(x_0,y_0) + iL_gu^{(1)}_{2k}(x_0,y_0)\\ L_g\widetilde{u}^{(1)}_{1k}(x_0,y_0) + iL_g\widetilde{u}^{(1)}_{2k}(x_0,y_0)\end{pmatrix}\\ \gamma_k(L_gx_0,L_gy_0) = L_g^{*-1}\gamma_k(x_0,y_0), k = 1,...,n\end{gathered} \tag{3.3}$$

and for the range of operators F_{k_x}', F_{k_y}' one has

$$R(F_{k_x}'(0,0,L_g x_0, L_g y_0, \lambda_0)) = R(K_g F_{k_x}'(0,0,x_0,y_0,\lambda_0)L_g^{-1}) = K_g R(F_{k_x}'(0,0,x_0,y_0,\lambda_0)),$$
$$R(F_{k_y}'(0,0,L_g x_0, L_g y_0, \lambda_0)) = R(K_g F_{k_y}'(0,0,x_0,y_0,\lambda_0)L_g^{-1}) = K_g R(F_{k_y}'(0,0,x_0,y_0,\lambda_0)).$$

Then for the kernel of adjoint operator

$$\mathcal{N}(\mathfrak{B}_0^*) = \mathcal{N}(\mathfrak{B}_0^*(x_0,y_0)) = \mathcal{N}\left(\mathbf{B}_0^*(x_0,y_0) + \alpha \mathbf{A}_0^*(x_0,y_0)\frac{d}{d\tau}\right) =$$
$$= \operatorname{span}\left\{\begin{pmatrix} v_{1k}^{(1)} + i v_{2k}^{(1)} \\ \widetilde{v}_{1k}^{(1)} + i\widetilde{v}_{2k}^{(1)} \end{pmatrix} e^{i\tau}, \begin{pmatrix} v_{1k}^{(1)} - i v_{2k}^{(1)} \\ \widetilde{v}_{1k}^{(1)} - i\widetilde{v}_{2k}^{(1)} \end{pmatrix} e^{-i\tau}\}\right\}_{k=1}^{n} \Rightarrow$$
$$\mathcal{N}(\mathbf{B}_0^*(L_g x_0, L_g y_0) + \alpha \mathbf{A}_0^*(L_g x_0, L_g y_0)\frac{d}{d\tau}) = \operatorname{span}\left\{K_g^{*-1}\psi_k, K_g^{*-1}\overline{\psi}_k\right\}_{k=1}^{n}$$

Analogously to [2], [7] it can be proved that the elements of the ordered by increasing lengths GJChs of the operator-function $\mathfrak{B}_0 - \mu\mathfrak{A}_0 = \mathfrak{B}_0(x_0,y_0) - \mu\mathfrak{A}_0(x_0,y_0)$ and biorthogonal to them systems are transformating according with the formulae

$$\begin{aligned}
&\varphi_k^{(s)}(L_g x_0, L_g y_0) = L_g \varphi_k^{(s)}(x_0,y_0) = [L_g(u_{1k}^{(s)}(x_0,y_0) + i u_{2k}^{(s)}(x_0,y_0)), L_g(\widetilde{u}_{1k}^{(s)}(x_0,y_0) + i\widetilde{u}_{2k}^{(s)}(x_0,y_0))]^T,\\
&\psi_k^{(s)}(L_g x_0, L_g y_0) = K_g^{*-1}\psi_k^{(s)}(x_0,y_0) =\\
&= [K_g^{*-1}(v_{1k}^{(s)}(x_0,y_0) + i v_{2k}^{(s)}(x_0,y_0)), K_g^{*-1}(\widetilde{v}_{1k}^{(s)}(x_0,y_0) + i\widetilde{v}_{2k}^{(s)}(x_0,y_0))]^T,\\
&\gamma_k^{(s)}(L_g x_0, L_g y_0) = L_g^{*-1}\gamma_k^{(s)}(x_0,y_0),\ z_k^{(s)}(L_g x_0, L_g y_0) = K_g \gamma_k^{(s)}(x_0,y_0)
\end{aligned} \tag{3.4}$$

Lemma 3.1. *Introduced in Lemma 2.1 projectors $\mathbf{P}(x_0,y_0)$ and $\mathbf{Q}(x_0,y_0)$ satisfy intertwining properties*

$$\begin{aligned}
\mathbf{P}(L_g x_0, L_g y_0) = L_g \mathbf{P}(x_0,y_0)L_g^{-1} \quad &or \quad L_g\mathbf{P}(x_0,y_0) = \mathbf{P}(L_g x_0, L_g y_0)L_g\\
\mathbf{Q}(L_g x_0, L_g y_0) = K_g \mathbf{P}(x_0,y_0)K_g^{-1} \quad &or \quad K_g\mathbf{P}(x_0,y_0) = \mathbf{P}(L_g x_0, L_g y_0)K_g
\end{aligned} \tag{3.5}$$

and generates the expansions (2.8) *in direct sums. Moreover the bases in the zero-subspaces $\mathcal{N}(\mathfrak{B}_0)$ and $\mathcal{N}(\mathfrak{B}_0^*)$ and respectively in the root-subspaces $\widetilde{\mathcal{H}}^{2K}(x_0,y_0)$ and $\widetilde{\mathcal{H}}_{2K}(x_0,y_0)$ can be chosen so that the following relations would be satisfied:*

$$\begin{aligned}
&\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{2K}(L_g x_0, L_g y_0) \dot{+} \widetilde{\mathcal{H}}^{\infty-2K}(L_g x_0, L_g y_0),\\
&\widetilde{\mathcal{H}}^{2K}(L_g x_0, L_g y_0) = L_g\widetilde{\mathcal{H}}^{2K}(x_0,y_0),\ \widetilde{\mathcal{H}}^{\infty-2K}(L_g x_0, L_g y_0) = L_g\widetilde{\mathcal{H}}^{\infty-2K}(x_0,y_0),\\
&\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{2K}(L_g x_0, L_g y_0) \dot{+} \widetilde{\mathcal{H}}_{\infty-2K}(L_g x_0, L_g y_0),\\
&\widetilde{\mathcal{H}}_{2K}(L_g x_0, L_g y_0) = L_g\widetilde{\mathcal{H}}_{2K}(x_0,y_0),\ \widetilde{\mathcal{H}}_{\infty-2K}(x_0,y_0) = L_g\widetilde{\mathcal{H}}_{\infty-2K}(x_0,y_0),
\end{aligned} \tag{3.6}$$

The proof follows from the formulae (3.3), (3.4).

T h e o r e m 3.1. ***(Group symmetry inheritance theorem.)*** *A.Lyapounov* (2.11) *and E.Schmidt* (2.13) *BEqRs inherit the group symmetry of the original system* (2.1)

$$\begin{aligned}
\mathbf{f}(L_g x_0, L_g y_0, L_g\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon) &= \mathbf{f}((L_g x_0, L_g y_0, \mathcal{V}(L_g x_0, L_g y_0,\xi,\overline{\xi}),\mu,\varepsilon) =\\
&= K_g\mathbf{f}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\mathbf{t}(L_g x_0, L_g y_0, L_g\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon) &= \mathbf{t}((L_g x_0, L_g y_0, \mathcal{V}(L_g x_0, L_g y_0,\xi,\overline{\xi}),\mu,\varepsilon) =\\
&= K_g\mathbf{t}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon).
\end{aligned} \tag{3.8}$$

Proof is essentially uses three-canonicity of the Jordan sets of the linear by μ operator-function $\mathfrak{B}_0 - \mu\mathfrak{A}_0$. According to (3.6) write the equation (2.7) in the bifurcation point $(L_g x_0, L_g y_0)$ in projections on the root-subspaces

$$\begin{aligned}
&[I - \mathbb{Q}(L_g x_0, L_g y_0)]\mathfrak{B}_0(L_g x_0, L_g y_0) = \\
&= [I - \mathbb{Q}(L_g x_0, L_g y_0)]\{\mu\mathfrak{A}_0(L_g x_0, L_g y_0)[...] + \mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}[...], [...], \mu, \varepsilon)\}, \\
&0 = \mathbb{Q}(L_g x_0, L_g y_0)\mathfrak{B}_0(L_g x_0, L_g y_0)\mathcal{V}(L_g x_0, L_g y_0, \xi, \overline{\xi}) - \\
&-\mathbb{Q}(L_g x_0, L_g y_0))\{\mu\mathfrak{A}_0(L_g x_0, L_g y_0) + \mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}[...], [...], \mu, \varepsilon)\}, \\
&[\ldots] = \widetilde{\mathcal{U}} + \mathcal{V}(L_g x_0, L_g y_0, \xi, \overline{\xi}).
\end{aligned}$$

For the restriction $\widehat{\mathfrak{B}_0}(x_0, y_0) = [I - \mathbb{Q}(x_0, y_0)]\mathfrak{B}_0(x_0, y_0)[I - \mathbb{P}(x_0, y_0)]$ of the operator $\mathfrak{B}_0(x_0, y_0)$ on the space $\widetilde{\mathcal{H}}^{\infty-2K}(x_0, y_0)$ the following symmetry relation is realized

$$\begin{aligned}
&K_g\widehat{\mathfrak{B}_0}(x_0, y_0) = K_g[I - \mathbb{Q}(x_0, y_0)]\mathfrak{B}_0(x_0, y_0)[I - \mathbb{P}(x_0, y_0)] \overset{(3.2),(3.5)}{=} \\
&\overset{(3.2),(3.5)}{=} [I - \mathbb{Q}(L_g x_0, L_g y_0)]\mathfrak{B}_0(L_g x_0, L_g y_0)L_g[I - \mathbb{P}(L_g x_0, L_g y_0)] \overset{(3.5)}{=} \\
&\overset{(3.5)}{=} [I - \mathbb{Q}(L_g x_0, L_g y_0)]\mathfrak{B}_0(L_g x_0, L_g y_0)[I - \mathbb{P}(L_g x_0, L_g y_0)]L_g = \\
&= \widehat{\mathfrak{B}_0}(L_g x_0, L_g y_0)L_g
\end{aligned}$$

Then the application K_g^{-1} to the first equation of the system gives

$$\begin{gathered}
K_g^{-1}\widehat{\mathfrak{B}_0}(x_0, y_0)L_g^{-1}\widetilde{\mathcal{U}} = \widehat{\mathfrak{B}_0}(x_0, y_0)L_g^{-1}\widetilde{\mathcal{U}} = \\
= K_g^{-1}[I - \mathbb{Q}(L_g x_0, L_g y_0)]\{\mu\mathfrak{A}_0(L_g x_0, L_g y_0)[...] + \mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}[...], [...], \mu, \varepsilon)\} \overset{(3.5)}{=} \\
\overset{(3.5)}{=} [I - \mathbb{Q}(L_g x_0, L_g y_0)]L_g^{-1}\{\mu\mathfrak{A}_0(L_g x_0, L_g y_0)[...] + \mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}[...], [...], \mu, \varepsilon)\} \overset{(3.5)}{\Rightarrow} \\
\overset{(3.5)}{\Rightarrow} \widehat{\mathfrak{B}_0}(x_0, y_0)L_g^{-1}\widetilde{\mathcal{U}} = [I - \mathbb{Q}(x_0, y_0)]\{\mu\mathfrak{A}_0[L_g^{-1}\widetilde{\mathcal{U}} + \mathcal{V}(x_0, y_0, \xi, \overline{\xi}) + \\
+\mathcal{R}(x_0, y_0, \tfrac{d}{d\tau}[L_g^{-1}\widetilde{\mathcal{U}} + \mathcal{V}(x_0, y_0, \xi, \overline{\xi})], [L_g^{-1}\widetilde{\mathcal{U}} + \mathcal{V}(x_0, y_0, \xi, \overline{\xi})], \mu, \varepsilon)\}
\end{gathered}$$

According to implicit operators theorem we find the unique solution of the last equation in the form $L_g^{-1}\widetilde{\mathcal{U}} = \mathcal{V}(x_0, y_0, \mathcal{V}(x_0, y_0, \xi, \overline{\xi})\mu, \varepsilon))$, the substitution of which into the second equation of the system gives A.Lyapounov BEqR in the point $(L_g x_0, L_g y_0)$ and its group symmetry (2.11)

$$\begin{gathered}
\mathbf{f}(L_g x_0, L_g y_0, \mathcal{V}(x_0, y_0, \xi, \overline{\xi}), \mu, \varepsilon) = \mathbb{Q}(L_g x_0, L_g y_0)\mathfrak{B}_0(L_g x_0, L_g y_0)\mathcal{V}(L_g x_0, L_g y_0, \xi, \overline{\xi}) - \\
-\mathbb{Q}(L_g x_0, L_g y_0)\{\mu\mathfrak{A}_0(L_g x_0, L_g y_0)[L_g\mathcal{U} + \mathcal{V}(L_g x_0, L_g y_0, \xi, \overline{\xi})] + \\
+\mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}[...].[...], \mu, \varepsilon)\} \overset{(3.2),(3.5)}{=} K_g\mathbb{Q}(x_0, y_0)\mathfrak{B}_0(x_0, y_0)\mathcal{V}(x_0, y_0, \xi, \overline{\xi}) - \\
-K_g\mathbb{Q}(x_0, y_0)\{\mu\mathfrak{A}_0(x_0, y_0)[\mathcal{U}(x_0, y_0) + \mathcal{V}(x_0, y_0, \xi, \overline{\xi})] + \mathcal{R}(x_0, y_0, \tfrac{d}{d\tau}[...].[...], \mu, \varepsilon)\} = \\
= K_g\mathbf{f}(x_0, y_0, \xi, \overline{\xi}, \mu, \varepsilon)
\end{gathered}$$

For the proof (3.8) write the equation (2.7) in the bifurcation point $(L_g x_0, L_g y_0)$ in the form of the system

$$\begin{gathered}
\widetilde{\mathfrak{B}}_0(L_g x_0, L_g y_0)L_g\mathcal{Y} = \mu\mathfrak{A}_0(L_g x_0, L_g y_0)L_g\mathcal{Y} + \mathcal{R}(L_g x_0, L_g y_0, \tfrac{d}{d\tau}L_g\mathcal{Y}, \mathcal{Y}, \mu, \varepsilon) + \\
+\sum_{i=1}^{n}[\xi_i z_i^{(1)}(L_g x_0, L_g y_0) + \overline{\xi}_i\overline{z}_i^{(1)}(L_g x_0, L_g y_0)], \\
\xi_i = \langle\langle L_g\mathcal{Y}, \gamma_j^{(1)}(L_g x_0, L_g y_0)\rangle\rangle,\ \overline{\xi}_i = \langle\langle L_g\mathcal{Y}, \overline{\gamma}_j^{(1)}(L_g x_0, L_g y_0)\rangle\rangle.
\end{gathered}$$

Setting $L_g\mathcal{Y} = \widetilde{w} + L_g\mathcal{V}(x_0, y_0, \xi, \overline{\xi}) = \widetilde{w} + \mathcal{V}(L_g x_0, L_g y_0, \xi, \overline{\xi})$ by virtue of group symmetry of the operators $\widetilde{\mathfrak{B}}_0$ $K_g\widetilde{\mathfrak{B}}_0(x_0, y_0)h = \widetilde{\mathfrak{B}}_0(L_g x_0, L_g y_0)L_g h, \mathfrak{A}_0, K_g\mathfrak{A}_0(x_0, y_0)h = \mathfrak{A}_0(L_g x_0, L_g y_0)L_g h$, and $\mathcal{R}$, (see auxiliary constructions) one has $\widetilde{\mathfrak{B}}_0(L_g x_0, L_g y_0)\widetilde{w} = K_g\widetilde{\mathfrak{B}}_0(x_0, y_0)L_g^{-1}\widetilde{w} =$

$K_g\{\mu\mathfrak{A}_0(x_0,y_0)[L_g^{-1}\widetilde{w} + \mathcal{V}(x_0,y_0,\xi,\overline{\xi})] + \mathcal{R}(x_0,y_0,\frac{d}{d\tau}[L_g^{-1}\widetilde{w} + \mathcal{V}(x_0,y_0,\xi,\overline{\xi}), L_g^{-1}\widetilde{w} + \mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)\}$ whence it follows $L_g^{-1}\widetilde{w} = w(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)$ or $\widetilde{w} = L_g w(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)$. Then from the second equation of the system E.Schmidt BEqR (2.13) in the point (L_gx_0, L_gy_0) and its group symmetry follows $\mathbf{t}(L_gx_0,L_gy_0,L_g\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon) = \mathbf{P}(L_gx_0,L_gy_0)\widetilde{w} \overset{(3.5)}{=} L_g\mathbf{P}(x_0,y_0)w(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon) = L_g\mathbf{t}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon)$.

4. Basic Results

Theorem 4.1. ***(Implicit operators theorem under group symmetry conditions.)*** *Let under continuous group symmetry conditions* (3.1) *of the system* (2.1) *the requirements* $(\mathbf{c_1})$ – $(\mathbf{c_3})$ *are realized, in the condition* $(\mathbf{c_2})$ $\kappa = n$ *and* $G_s(a), s < l$ *is the normal divisor of* $G_l(a)$ *with the relevant ideal* $T^s_{g(a)}$ *of generators. For the operator-function* $\mathfrak{B}_0 - \mu\mathfrak{A}_0$ *with Fredholm operator* $\mathfrak{B}_0$ *always can be chosen the complete three-canonical GJS to elements of* $\mathcal{N}(\mathfrak{B}_0)$. *Then there exists the continuous function* $\mathcal{V}(x_0,y_0,\xi,\overline{\xi},\mu,\varepsilon) = \mathcal{V}(x_0,y_0,\xi,\overline{\xi}) + \mathcal{U}(x_0,y_0,\mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon) : T^{2n}_{g(a)}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \times (-\delta,\delta) \to \widetilde{\mathcal{H}}$, *invariant with respect to the factor-group* $G_\kappa = G_n = G_l/G_s$ *on* $T^{2n}_{g(a)}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, *such that for the nonlinear operator* $\mathcal{F}$

$$\mathcal{F}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathcal{V}(x_0,y_0,\xi,\overline{\xi}),\mu,\varepsilon\right) = 0, \quad for \quad \mathcal{V}(x_0,y_0,\xi,\overline{\xi}) \in T^{2n}_{g(a)}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \ |\varepsilon| < \delta, \qquad (4.1)$$

where the nonlinear operator $\mathcal{F}$ *is defined by the equation* (2.7).

Corollary. Theorem 4.1 is true for semisimple bifurcation points, i.e. at the absence of GJS. Then here we have BEq.

Definition 4.1. [14] *BEqR* (2.11) *(respect.* (2.13)*) is the BEqR of potential type* ***(A)*** *if in a neighborhood of the point* $(x_0, y_0; 0)$ *for the vector* $\mathfrak{f}(x,y,v(x,y,\xi,\overline{\xi}),\mu,\varepsilon) = (f_{11},\overline{f}_{11},\ldots,f_{1p_1},\overline{f}_{1p_1},\ldots,f_{n1},\overline{f}_{n1},\ldots,f_{np_n},\overline{f}_{np_n})$ *the equality*

$$\mathfrak{f}(x,y,\mathcal{V}(x,y,\xi,\overline{\xi}),\mu,\varepsilon) = d\cdot grad_{x,y}U(x,y,\xi,\overline{\xi},\mu,\varepsilon) \qquad (4.2)$$

is satisfied and potential type ***(B)****, when in in a neighborhood of the point* $(x_0, y_0; 0)$

$$\mathfrak{f}(x,y,\mathcal{V}(x,y,\xi,\overline{\xi}),\mu,\varepsilon) = grad_{x,y}U(x,y,\xi,\overline{\xi},\mu,\varepsilon)\cdot d \qquad (4.3)$$

is satisfied where d *is an invertible operator. Then the functional* $U(x,y,\xi,\overline{\xi}),\mu,\varepsilon)$ *is the potential of BEqR* (2.11) *(resp.* (2.13)*) and the operator* $\mathfrak{f}$ *(resp.* $\mathfrak{t}$*) is pseudogradient of the functional* U.

In the case of BEqR (2.11) or (2.13) potentiality type $(\mathbf{A})$ in previous our articles [15], [16], [2] the necessary and sufficient condition of the L_g-invariance of the potential U is established. This is the equality $L_g^* d^{-1} K_g = d^{-1}$ for the A.Lyapounov BEqR (2.11) and $L_g^* d^{-1} L_g = d^{-1}$ for the E.Schmidt BEqR (2.13) . Also it is proved that the pseudogradient f (resp. t) of the L_g-invariant functional U is (L_g, K_g)-(resp. (L_g, L_g)-) equivariant in the sense (2.11) (resp. (2.13)) together with cosymmetric identities for the BEqR left-hand-sides.

In the same manner the respective results for BEqRs of the type $(\mathbf{B})$ can be proved. These are the equality $K_g^{-1} = L_g^*$ (resp. $L_g^{-1} = L_g^*$) for the BEqR (2.11) (BEqR (2.13)) and the relevant equivariance results with cosymmetric identities.

T h e o r e m 4.2. ***(BEqR reduction.)*** *Let in suppositions* $(\mathbf{c_1})$ – $(\mathbf{c_3})$ *A.Lyapounov BEqR (E.Schmidt BEqR) is potential type* ***(A)*** *or* ***(B)****, its potential* $U(x,y,\xi,\overline{\xi}),\mu,\varepsilon)$ *is invariant of the representation* $L_{g(a)}$ *of the group* $G_l(a)$ *and belongs to the class* C^2 *in some neighborhood of the bifurcation point* $(x_0,y_0;0)$ *,* s *– the dimension of stationary subgroup of the element* (x_0,y_0) *and* $\kappa = l - s > 0$ *. Then:*

1. *if* $\kappa = n$ *, then for all* $(\xi(\varepsilon),\overline{\xi}(\varepsilon),\mu,\varepsilon)$ *or* $(\mathcal{V}(x_0,y_0,\xi(\varepsilon),\overline{\xi}(\varepsilon),\mu,\varepsilon)$ *in some neighborhood of zero in* Ξ^{2n} *BEqR* (2.11) *(respect.* (2.12)*) is identically fulfilled.*

2. *if* $\kappa < n$ *and* $n \geq 2$ *, then the partial reduction of BEqR takes place: at the accepted in the condition* $(\mathbf{c_2})$ *agreement on the basic elements enumeration in* $\widetilde{\mathcal{H}}^{2K}$ *the first* $K_\kappa = p_1 + \ldots + p_\kappa$ *equations are linear combinations of the others* $p_{\kappa+1} + \ldots + p_n$ *.*

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Dynamic bifurcation problems with E.Schmidt spectrum in the linearization under group symmetry conditions

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Abstract. Results of the articles [1], [2] for stationary problems of branching theory with E. Schmidt spectrum in the linearization are transformating on dynamic bifurcation problems on E. Schmidt spectrum. On the base of general theorem on the group symmetry theorem on the group symmetry of nonlinear problems inheritance by the relevant branching equations and branching equations in the root-subspaces (BEqRs), moving along bifurcation point trajectory implicit operators theorem under group symmetry conditions and theorem on BEqRs reduction by the number of equations in the case of variational BEqRs are proved. Terminology and notations of the works [3]– [6] are used.

Key Words: Dynamic bifurcation problems, Poincarè-Andronov-Hopf bifurcation; E. Schmidt spectrum, group symmetry, G-invariant implicit operator theorem, variational type branching equations in the root-subspaces.

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