Polynomials based methods for linear nonconstant coefficients eigenvalue problems

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Annotation. A method based on generalized Jacobi polynomials is proposed to solve the eigenvalue problem governing the Lyapunov stability of the mechanical equilibria of certain fluids occurring in complex circumstances. Two concrete natural convection problems of great interest from the applications point of view are numerically investigated. Fairly accurate approximations of the lower part of the spectrum are given in comparison with other numerical evaluations existing in the literature.
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1. Introduction

Spectral methods have been applied with great success to various physical problems from science and engineering for which the evolution of perturbations is governed by linear or nonlinear eigenvalue problems [1], [5], [6],[11], [13]. The main reason of their extensive use is the high accuracy of these methods and the fact that the expansion functions usually have a basic property: they are easy to evaluate either Fourier series based on trigonometric functions or polynomials expansions are considered.

Usually the linearization process in a hydrodynamic stability problem increases the conditions number of the problem making the solution more sensitive to small perturbations. The resulting eigenvalue problem depends on the spectrum of the operator obtained by the linearization of the mapping that define the initial and boundary conditions eigenvalue problem.

The use of classical Jacobi polynomials as trial functions in weighted Galerkin-type methods is motivated by the fact that spectral approximations of the eigenfunctions in an eigenvalue problems is usually considered as a finite expansion of eigenfunctions of a singular Sturm-Liouville problem and Jacobi polynomials are in fact eigenfunctions of such a problem [2]. For problems with singular or degenerated coefficients or some problems on infinite intervals the Jacobi polynomials are of great interest [8], [9].

A dual-Petrov-Galerkin method based on Jacobi polynomials was introduced in [10] for third and higher odd-order differential equations. It was proven that their use simplify the numerical analysis of the spectral approximations. The Chebyshev and Legendre polynomials (regained as particular cases of Jacobi polynomials) have been widely used in the literature with good results in [4], [6],[12].

The main objective of the paper is to point out the major dependence of the approximations properties of spectral methods on the choice of the basis functions. General Jacobi polynomials based spectral methods for differential equations were not widely used in mathematical physics problems. In [2] Jacobi-Galerkin methods for fourth - order equations in two dimensions are

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considered proving that for a suitable selection of the parameters α, β the resulting systems of equations to be solved are diagonals, simplifying the numerical procedure.

As examples of applications, we considered two particular physical problems governed by high-order differential equations with variable coefficients. The eigenvalue problem governing the stability of the fluid motion in both cases has a general formulation

$$\begin{cases} A\mathbf{U} = Ra \cdot B\mathbf{U}, \text{ in } \Omega \\ C\mathbf{U} = 0, \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

with A, B nonconstant elements matrix high order differential operators and C a linear differential operator defined on the boundary $\partial\Omega$ of the domain of definition Ω by

$$C(\mathbf{U}, D\mathbf{U}) = 0. \tag{1.2}$$

This paper is organized as follows. In this section we introduce some generalized Jacobi polynomials [2] and motivate their use for the approximation of the orthogonal projection on some Hilbert space for our type of problems. The second section is devoted to the analytical and numerical applications of the proposed method to the particular physical problems. Some final remarks are given in the conclusion section of the paper.

Let us introduce the differential operators A, B from (1.1)-(1.2). In both cases, the general formulation of our problems leads to the two-point boundary value problem

$$\begin{cases} (D^2 - a^2)^2 W = f(z)\Theta, \\ (D^2 - a^2)\Theta = -a^2 Rag(z)W, \end{cases} \tag{1.3}$$

so $A = \begin{pmatrix} (D^2 - a^2 I)^2 & f(z) \\ O & (D^2 - a^2 I) \end{pmatrix}$ and $B = \begin{pmatrix} O & O \\ -a^2 g(z) & O \end{pmatrix}$. The boundary conditions defining the operator C have the form

$$W = DW = \Theta = 0 \text{ at } z = 0, 1. \tag{1.4}$$

Here f, g are two indefinitely derivable functions characterizing the basic flow, W, Θ the amplitudes of the velocity and the temperature perturbation fields, (W, Θ) representing the corresponding eigenvector for the eigenvalue Ra.

A suitable approach imply a transformation of the physical domain onto the standard interval of definition of Jacobi polynomials, i.e. $x = 2z - 1$, such that the boundary conditions are written at -1 and 1

$$W = DW = \Theta = 0 \text{ at } x = -1, 1. \tag{1.5}$$

The weighted residual method proposed here imply a spectral expansion of each component of the eigenvector using combination of generalized Jacobi polynomials functions that satisfy the boundary conditions (1.4).

Let us recall that the Jacobi polynomials $P_n^{\alpha,\beta}(x)$, $n > 0$ are defined by the Rodrigues formula [9]

$$P_n^{\alpha,\beta} = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}], \tag{1.6}$$

with $D = \dfrac{d}{dx}$ and α, β two complex parameters.

The classical Jacobi polynomials associated with the real parameters $\alpha, \beta > -1$ are a sequence of orthogonal polynomials, i.e.

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx = \gamma_n^{\alpha,\beta} \delta_{n,m} \tag{1.7}$$

with $\omega^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ the Jacobi weight function, $\delta_{n,m}$ the Kronecker symbol and

$$\gamma_n^{\alpha,\beta}=\frac{2^{\lambda}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\lambda)\Gamma(n+1)\Gamma(n+\lambda)},\ \lambda=\alpha+\beta+1.$$

Jacobi polynomials can also be defined using (1.6) for $\alpha,\beta<-1$. However, the main property used in numerical applications, the orthogonality in $L^2_{\omega^{\alpha,\beta}}$ for all α, β of these polynomials it is no longer valid.

Following [11] to account for the homogeneous boundary conditions, let us define the space P_N of all polynomials of degree less or equal to N, $N=1,2,\dots$. We are interested in the construction of an approximation space of the form $\mathcal{M}=\{v\in P_N|v=Dv=0 \text{ at } x=\pm1\}$. Let us introduce the functions $\varphi_k\in\mathcal{M}$, $k=1,2,\dots$, [9]

$$\varphi(x)=(1-x^2)^2\cdot P_k^{\alpha,\beta}(x),\quad k=1,\dots,N$$

which fulfills the boundary conditions (1.4). Using the properties of the Jacobi polynomials it is easy to verify that the functions $\varphi_k(x)$, $0<k<N-4$, are linearly independent and the dimension of the corresponding generated space $\mathcal{N}=span\{\varphi_0(x),\dots,\varphi_{N-4}(x)\}$ is equal to $N-3$. In fact, these function can be viewed as generalized Jacobi polynomials [2] since we can write

$$\varphi(x)=(1-x)^2(1+x)^2P_n^{\alpha,\beta}(x)=P_n^{\alpha',\beta'}(x).$$

The above expression point out that a certain type of indexes for generalized Jacobi polynomials must be used. In fact, in [12] it was proven that when developing and analyzing Chebyshev spectral methods for boundary value problems the generalized Jacobi polynomials with indexes $(-1/2-k,-1/2-l)$, $k,l\in\mathbb{Z}$ are the most convenient choice.

Let us introduce the expansion series $W=\sum\limits_{k=0}^{N-4}W_k\varphi_k(x)$, $\Theta=\sum\limits_{k=0}^{N-4}\Theta_k\varphi_k(x)$, with $W_k=(W,\varphi_k(x))$ and $\Theta_k=(\Theta,\varphi_k(x))$ and the scalar product $(\cdot,\cdot)$ taken with respect to the Hilbert space $L^2_{\omega^{\alpha,\beta}}$.

Replacing the expansion functions in the system (1.3) and imposing the condition that the equations in (1.3) be orthogonal on $\{\varphi_i\}_{i=1,\dots,N-4}$ leads to an algebraic system in the unknown coefficients W_k, Θ_k. Not all these coefficients vanish so the condition that the determinant of the algebraic system be equal to zero leads to the secular equation which gives us the critical values of the Rayleigh number.

The formulas for the Jacobi coefficients of all the derivatives of the functions occurring in (1.3) can be found in [2].

2. Particular physical convection problems

We considered two concrete physical cases: one concerning a convection problem for a variable gravity field [6] and the other one for a convection problem with an internal heat source [4].

2.1. A convection problem with variable gravity field

The convection problem investigated in this section arises in a horizontal layer of fluid heated from below for a variable gravity field. The gravity field varying across the layer can be considered linear or not [5], [6]. Here our investigation concerns only a linearly decreasing gravity field orthogonal on the fluid layer and assumed to depend on the vertical coordinate z only [13]. The linear stability against normal mode perturbations is governed by a two-point problem for ordinary differential equations of the form (1.3) with $f(z) = 1 - \varepsilon z$ and $g(z) = 1$, with ε the scale parameter for the variable gravity field. The analytical and numerical investigation of the problem was also performed in [3], using a shifted Legendre polynomials based method for the case of linear stability of the mechanical equilibrium of the fluid layer. In this paper we apply the above described method for various cases of the parameters defining the Jacobi polynomials. The chosen basis of expansion functions leads to sparse matrices, with banded submatrices whose size are equal to the number of generalized Jacobi polynomials used in the expansions.

Numerical evaluations of the Rayleigh number for various values of the wavenumber and various linear decreasing gravity fields are presented in Table 1 in comparison with previous results obtained also by us for either trigonometric expansion functions or shifted Legendre polynomials. The following physical conclusion is pointed out: the stability domain increases as the gravity field is linearly decreasing across the layer.

ε	a^2	$Ra_{\alpha,\beta=-1/2}$	$Ra_{\alpha,\beta=0}$	$Ra_{\alpha,\beta=1/2}$	Ra_{trig}[6]	Ra_{SLP}[6]
0	9.711	1730.0	1748.5	1743.9	1715.079356	1749.975727
0.01	9.711	1738.8	1757.2	1752.8	1723.697848	1758.769253
0.2	9.711	1922.2	1942.3	1937.5	1905.643719	1944.243122
0.2	12.0	1951.3	1969.6	1965.1	1937.927940	1977.079049
0.2	14.5	2037.1	2053.9	2049.9	2026.289430	3475.507241
1	10.0	3434.5	3470.8	3461.8	3431.318766	3475.507241

Table 1. Numerical evaluations of the Rayleigh number for various values of the parameters ε and a and various parameters α, β.

It is clear that for the same small value of the spectral parameter, $N = 3$, the numerical results obtained here are similar, but not the best. However, we remarked that the necessary computational time is significantly reduced in this case.

In Table 2 the numerical evaluations of the first and second eigenvalue, respectively, are presented for increasing values of the spectral parameter N. The results are obtained for the classical Rayleigh-Bénard convection and they show a quite good agreement with previous existing values for the critical Rayleigh number.

N	Ra_1	Ra_2
2	1790.0	27286
5	1757.2	25801
10	1729.7	25443
12	1726.6	25326
14	1724.6	25306
15	1724.5	25272

Table 2. Numerical evaluations of the first and the second eigenvalue for $\alpha = \beta = 0$, $a^2 = 9.711$, $\varepsilon = 0$ for various values of the spectral parameter N.

The decreasing of both eigenvalues to well known numerical values with an increasing N marks a numerical convergence of the algorithm (Fig. 1 a), b)).

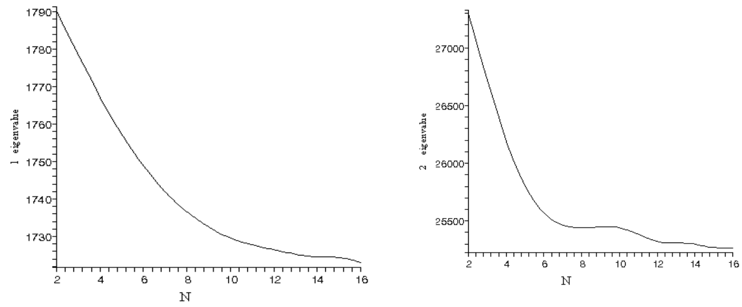

Fig. 1. The decreasing values of the first and the second eigenvalue Ra for an increasing spectral parameter N.

2.2. A convection problem with an internal heat source

The motion in the atmosphere or mantle convection in the Earth are two among phenomena of natural convection induced by internal heat sources. Natural convection occurring in industrial devices by internal heating [14] is another example for which an investigation of the effect of internal heat generation on fluids flows is needed. All these motions bifurcate from the conduction state as a result of its loss of stability. In spite of their importance, due to the occurrence of variable coefficients in the nonlinear partial differential equations governing the evolution of perturbations around the basic equilibrium, few systematic studies were performed. Most of the investigations only consider the much simpler case of uniform heat generation. In [15] experimental investigations were carried out pointing out that a dilation of convection cells occurs with an increasing rate of internal heat generation. The physical model of natural convection in the presence of an uniform internal heat source, investigated in this paper, concerning a horizontal layer of viscous incompressible fluid with constant viscosity and thermal conductivity coefficients ν and k, was also treated in [16]. Veltchiev [16] focussed on the vertical distribution of the total heat fluxes and their individual components for small and moderate supercritical Rayleigh number. An analytical investigation of the eigenvalue problem deduced by us in [4] was performed in [7]. Here we are concerned with the approximative numerical evaluations of the critical Rayleigh number at which the instability sets in. These results obtained for various types of polynomial approximations when generalized Jacobi polynomials are considered, are presented in comparison with the ones obtained by using other type of polynomials (Table 3).

The associated eigenvalue problem in a horizontal fluid layer bounded by two rigid walls, governing the stability of the basic motion against normal mode perturbations, deduced by us in [4], has the form from (1.3) with $f(z) = 1$ and $g(z) = 1 - N_{hc}z$. The eigenvalue is the Rayleigh number R and N_{hc} is a dimensionless parameter characterizing the heating (cooling) rate of the layer. Although it looks like a simple switch in the expression of the unknown functions f and g this new model support most of the times a different approach from the first benchmark model [7].

N_{hc}	a^2	$Ra_{\alpha,\beta=-1/2}$	$Ra_{\alpha,\beta=0}$	Ra_{hyp}[7]	Ra_{SLP}[5]
0	9.711	1730.2	1780.9	1708.54	1715.079324
1	9.711	1727.04	1745.3	1651.04	1711.742588
2	9.711	1717.5	1735.9	1609.12	1701.891.001
1	10.0	1727.1	1745.1	1651.1	1712.257687
4	10.0	1680.6	1696.4	1560.8	1664.341789
4	12.0	1699.4	1699.7	1739.2	1685.422373
10	9.0	1503.2	1524.6	1366.02	1482.527042
11	9.0	1468.0	1489.4	1366.05	1446.915467
12	9.0	1432.7	1454.2	1354.7	1411.401914

Table 3. Numerical evaluations of the Rayleigh number for various values of the parameters N_{hc} representing the heating (cooling rate) and a and various types of polynomials.

3. Conclusions

A family of generalized Jacobi polynomials with indexes (α, β) of the form $(-1/2-k, -1/2-k)$, $k \in \mathbb{Z}$ is proposed in order to solve a class of eigenvalue problems governing the linear stability of the mechanical equilibria of certain types of fluids motions. For boundary conditions corresponding to rigid boundary surfaces case, Fourier series based on the proposed generalized Jacobi polynomials basis led to a good numerical algorithm. In general, the method can be successfully used for spectral approximations of differential equations with suitable boundary conditions which are automatically satisfied by the expansion functions. Orthogonal families of generalized Jacobi polynomials can be constructed starting from the proposed one with a large applicability to solve partial or ordinary differential equations with constant or varying coefficients.

In spite of the existence of many theoretical bases, the complexity in reducing the computation of the critical Rayleigh number using spectral methods comes from the necessity that these functions satisfy all less simpler boundary conditions.

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