DOI 10.15507/2079-6900.28.202602.28-46
Original article

ISSN 2079-6900 (Print)
ISSN 2587-7496 (Online)

MSC2020 30B10, 37L05

On the Integrability of Dynamical Systems in the Space of Double-Sided Sequences

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Abstract. The article is devoted to the analysis of dynamical systems in a specially constructed countably-dimensional phase space, namely, in the space of double-sided sequences. It is not only a linear space but also a Banach algebra. Using the multiplication operation in this algebra, the article constructs two examples of dynamical systems in such a space; both systems allow exhaustive investigation. One dynamical system is concentrated, meaning that its dynamic variables depend only on time, while another system is distributed, meaning that its dynamic variables depend not only on time but also on a spatial variable. A concentrated system can be obtained in two ways. The first way is simplifying the Cauchy problem for the Kolmogorov-Petrovsky-Piskunov equation with a periodic initial condition. The second way involves deriving the evolution equation for the countably-dimensional energy in a countably-dimensional system of ordinary differential equations, which is the normal form for the analogue of the Andronov-Hopf bifurcation in the space of double-sided sequences. A distributed dynamic system arises as a result of the spatial discretization of a certain nonlinear integro-differential equation. In this paper general solutions of Cauchy problems for these dynamic systems are constructed using the method of generating functions in the form of Laurent series. The article also provides specific examples of exact solutions of Cauchy problems for these dynamic systems. It is shown how, by analogy with digital signal processing methods, these exact solutions generate a countable set of exact solutions for the dynamic systems under consideration.

Keywords: Fourier series, commutative algebra with unit, open ring, upsampling

For citation: *A. E. Rassadin.* On the Integrability of Dynamical Systems in the Space of Double-Sided Sequences. *Zhurnal Srednevolzhskogo matematicheskogo obshchestva.* 28:2(2026), 28–46. DOI: 10.15507/2079-6900.28.202602.28-46

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Оригинальная статья

УДК 517.9

Об интегрируемости динамических систем в пространстве двусторонних последовательностей

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Аннотация. Статья посвящена анализу динамических систем в специальным образом сконструированном счётномерном фазовом пространстве, а именно, в пространстве двусторонних последовательностей. Оно является не только линейным пространством, но и банаховой алгеброй. С помощью операции умножения в этой алгебре в статье строятся два примера динамических систем в таком пространстве, допускающих исчерпывающее исследование. Одна динамическая система является сосредоточенной, то есть её динамические переменные зависят только от времени, а другая динамическая система является распределённой, то есть её динамические переменные зависят не только от времени, но и от пространственной переменной. Сосредоточенная система может быть получена двумя способами. Первый способ состоит в упрощении задачи Коши для уравнения Колмогорова-Петровского-Пискунова с периодическим начальным условием. Второй способ заключается в выводе эволюционного уравнения для счётномерной энергии в счётномерной системе обыкновенных дифференциальных уравнений, являющейся нормальной формой для аналога бифуркации Андронова-Хопфа в пространстве двусторонних последовательностей. Распределённая динамическая система возникает в результате пространственной дискретизации некоторого нелинейного интегродифференциального уравнения. В данной работе общие решения задач Коши для этих динамических систем строятся с использованием метода производящих функций в форме ряда Лорана. В статье также приводятся конкретные примеры точных решений задач Коши для этих динамических систем. Показано, как по аналогии с методами цифровой обработки сигналов эти точные решения порождают счетное множество точных решений для рассматриваемых динамических систем.

Ключевые слова: ряды Фурье, коммутативная алгебра с единицей, открытое кольцо, прореживание

Для цитирования: Рассадин А. Э. Об интегрируемости динамических систем в пространстве двусторонних последовательностей // *Журнал Средневолжского математического общества.* 2026. Т. 28, № 2. С. 28-46. DOI: 10.15507/2079-6900.28.202602.28-46

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1. Introduction

Dynamical system is a mathematical model of the evolution of a real (physical, chemical, biological, economic, etc.) system, the states of which at any given time is uniquely determined by its initial state [1].

Let one remind strict definition of dynamical system with phase space $\mathbb{X}$ and continuous time t [2–4].

Definition 1.1. *A continuous mapping $\psi : \mathbb{X} \times \mathbb{R} \mapsto \mathbb{X}$ is called continuous dynamical system, if for all $x \in \mathbb{X}$ and all $t, s \in \mathbb{R}$ it obeys to the following group properties:*
i. $\psi(x, 0) = x$;
ii. $\psi(\psi(x, t), s) = \psi(x, t + s)$.

If phase space $\mathbb{X}$ of the system is infinite-dimensional Banach space over field $\mathbb{C}$, then this continuous dynamical system is called a distributed dynamical system, a lot of these systems in various applications being polynomially nonlinear.

If such Banach space possesses by unconditional Schauder basis, then input distributed dynamical system can be reduced to countable-dimensional system of ordinary differential equations (see, for instance, works [5–6] and references therein).

According to the article [7] let one demonstrate this procedure on example of the Cauchy problem for the Kardar–Parizi–Zhang equation:

$$\frac{\partial \hat{U}}{\partial t} = \frac{\partial^2 \hat{U}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \hat{U}}{\partial x}\right)^2, \quad \hat{U}(x, 0) = \hat{U}^0(x), \quad x \in \mathbb{R} \tag{1.1}$$

with periodic initial condition:

$$\hat{U}^0(x + 2\pi) = \hat{U}^0(x)\,. \tag{1.2}$$

This equation describes epitaxial growth of crystals.

Restriction (1.2) means that one can seek solution of equation (1.1) as the Fourier series:

$$\hat{U}(x, t) = \sum_{n=-\infty}^{+\infty} u_n(t)\, \exp(i\, n\, x), \quad u_{-n}(t) = u_n^*(t). \tag{1.3}$$

Substituting expansion (1.3) into input equation (1.1) one obtains the next countable-dimensional system of ordinary differential equations:

$$\dot{u}_n = -n^2 u_n - \frac{1}{2} \sum_{k=-\infty}^{+\infty} k\,(n-k)\, u_k\, u_{n-k}\,, \quad n \in \mathbb{Z}\,, \tag{1.4}$$

the Fourier amplitudes of initial condition (1.2) being initial conditions for system (1.4).

Carefully peering into system (1.4) we can observe that in this system set of independent variables $\{u_n\}_{n=-\infty}^{n=+\infty}$ may be considered as a vector of some linear space of the complex-valued double-sided sequence. Countable-dimensional system (1.4) of ordinary differential equations generates a flow in this space. However, the peculiarity of this example is that the right-hand side of this system includes both the components of this vector and their numbers. Other examples of such systems have the same properties [5–6].

The imposition of an exclusion on the dependence of the right-hand side of such system on the numbers of the vector components leads to the consideration of the following set of dynamical systems.

Definition 1.2. *Let $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots)$ be the complex-valued double-sided sequence. We shall say that u is an element of the linear* *space of complex-valued double-sided sequences* $\mathbb{L}$, *if for all* $z \in K$, *where* $K = \{z \in \mathbb{C} \mid r <| z |< R, r \geq 0, R \leq \infty\}$ *is some open ring, series* $\sum\limits_{n=-\infty}^{n=+\infty} u_n z^n$ *is the Laurent series for some analytical on* K *function* $U(z)$.

Definition 1.3. *Analytical on the ring* K *function*

$$U(z) = \sum_{n=-\infty}^{+\infty} u_n z^n \tag{1.5}$$

is called generating function for vector $u \in \mathbb{L}$.

If $u \in \mathbb{L}$ and $v \in \mathbb{L}$, then it is easy to find that product of corresponding them generating functions $U(z)$ and $V(z)$ is equal to:

$$U(z)\, V(z) = \sum_{n=-\infty}^{+\infty} (u \star v)_n \, z^n \,, \tag{1.6}$$

where $u \star v$ denotes double-sided sequence with components:

$$(u \star v)_n \; = \sum_{k=-\infty}^{+\infty} u_k \, v_{n-k} \,, \quad n \in \mathbb{Z}\,. \tag{1.7}$$

Formula (1.7) means that one can define product of two vectors from $\mathbb{L}$ without usage of generating functions.

It is well-known [8] that linear space $\mathbb{L}$ with multiplication (1.7) of two vectors is commutative algebra with unit $e = (\ldots, 0, 1, 0, \ldots)$. In particular $u \star e = e \star u = u$ for all $u \in \mathbb{L}$.

Let the elements of algebra $\mathbb{L}$ be functions of continuous time t: $u(t), v(t), w(t), \ldots \in \mathbb{L}$, then flows can be set on $\mathbb{L}$ using countable-dimensional systems of ordinary differential equations, the right-hand sides of which are constructed using addition and multiplication operations in the algebra $\mathbb{L}$.

For example, in work [9] the next system for vectors $v(t) \in \mathbb{L}$ and $w(t) \in \mathbb{L}$ was considered:

$$\dot v = \mu\, v - w - v \star (v \star v + w \star w), \quad \dot w = v + \mu\, w - w \star (v \star v + w \star w), \quad \mu > 0\,. \tag{1.8}$$

This system is countable-dimensional analog of the normal form of the Andronov–Hopf bifurcation, phase space of system (1.8) being $\mathbb{X} = \mathbb{L} \bigoplus \mathbb{L}$.

Further, the suggested construction can be generalized, namely, let the elements of algebra $\mathbb{L}$ be both functions of continuous time t and variable $x \in \mathbb{R}^m$: $u(x,t)$, $v(x,t)$, $w(x,t)$, $\ldots \in \mathbb{L}$, then distributed dynamical systems can be set on $\mathbb{L}$ using countable-dimensional systems of integro-differential equations or partial differential equations, the right-hand sides of which are constructed using both addition and multiplication operations in the algebra $\mathbb{L}$ and partial derivatives and integrals.

In particular, to illustrate this generalization the following Cauchy problem for vector $u(x,t) \in \mathbb{L}$ has been solved exactly in article [10]:

$$\frac{\partial u(x,t)}{\partial t} + \int\limits_{-\infty}^{+\infty} u(x-\xi,t) \star u(\xi,t)\; d\xi = 0, \quad u(x,0) = u^0(x) \in \mathbb{L}, \quad x \in \mathbb{R}\,. \tag{1.9}$$

The paper continues investigation of the integrability of dynamical systems in the space of double-sided sequences determined by Definition 1.2. The rest of this work is organized as follows: the next section deals with construction of general solution of countable-dimensional analog of the well-known logistic equation. Section 3 is devoted to the description in explicit form of a number of exact solutions of this nonlinear countable-dimensional system of ordinary differential equations. In section 4 general solution of one countable-dimensional system of nonlinear partial differential equations is found. Section 5 presents a number of exact solutions of this system of partial differential equations. Final section contains discussion of obtained results and of perspectives of further investigations.

2. General solution of the analog of the logistic model

Let us consider the next Cauchy problem for vector $u(t) \in \mathbb{L}$:

$$\dot{u} = u - u \star u, \quad u(0) = u^0 \in \mathbb{L}\,. \tag{2.1}$$

Equation (2.1) looks like the well-known logistic equation in which multiplication is replaced by an asterisk ("$\star$") denoting multiplication in $\mathbb{L}$ according to formula (1.7).

Using this formula one can rewrite equation (2.1) in components as countable-dimensional system of ordinary differential equations:

$$\dot{u}_n = u_n - \sum_{k=-\infty}^{+\infty} u_k\, u_{n-k}\,, \quad u_n(0) = u_n^0\,, \quad n \in \mathbb{Z}\,. \tag{2.2}$$

The system (2.1) (or (2.2)) may arise by means of three different ways.

First of all, it is easy to see that system (2.2) can be obtained from the Cauchy problem for the following integro-differential equation:

$$\frac{\partial \hat{U}(x,t)}{\partial t} = \hat{U}(x,t) - \int\limits_{-\infty}^{+\infty} \hat{U}(x-\xi,t)\, \hat{U}(\xi,t)\, d\xi, \quad \hat{U}(x,0) = \hat{U}^0(x), \quad x \in \mathbb{R}\,, \tag{2.3}$$

namely, discretization of straight line $\mathbb{R}$ with step δ generates from function $\hat{U}(x,t)$ countable set of functions $u_n(t) = \hat{U}(n\,\delta, t)$, $n \in \mathbb{Z}$. And after that change of integral in equation (2.3) by corresponding to it integral sum and further scaling $\delta\, u_n \to u_n$ ought to give rise to system (2.2) exactly.

Further, let us consider the next Cauchy problem for the Kolmogorov-Petrovskii-Piskunov equation [11]:

$$\frac{\partial \hat{U}}{\partial t} = D\,\frac{\partial^2 \hat{U}}{\partial x^2} + \hat{U}\,(1 - \hat{U})\,, \qquad \hat{U}(x,0) = \hat{U}^0(x)\,, \qquad x \in \mathbb{R}\,, \tag{2.4}$$

where D is a diffusion coefficient.

If initial condition for this Cauchy problem is periodic (1.2), then one ought to seek solution of equation (2.4) as the Fourier series (1.3).

Substituting expansion (1.3) into input equation (2.4) one obtains:

$$\dot{u}_n = -D\,n^2 u_n + u_n - \sum_{k=-\infty}^{+\infty} u_k\, u_{n-k}, \quad n \in \mathbb{Z}\,, \tag{2.5}$$

the Fourier amplitudes u_n^0 of initial condition

$$u_n^0 = \frac{1}{2\,\pi}\int\limits_0^{2\,\pi} \hat{U}^0(x)\exp(-i\,n\,x)\,dx \tag{2.6}$$

being initial conditions for system (2.5).

If the diffusion coefficient D is small: $0 < D << 1$, and values (2.6) vanishes under $|n| \geq \dfrac{1}{\sqrt{D}}$ as $o(n^{-2})$, that is if Fourier spectrum (2.6) of initial condition $\hat{U}^0(x)$ is concentrated in the strip with $|n| << \dfrac{1}{\sqrt{D}}$, then it is obvious that the solution of the Cauchy problem (2.5)-(2.6) is close to the solution of the Cauchy problem (2.2) with the same initial conditions at the initial stage of its temporal evolution.

At last let one construct from solutions $v(t) \in \mathbb{L}$ and $w(t) \in \mathbb{L}$ of the system (1.8) the following value:

$$u = \frac{v \star v + w \star w}{2}\,. \tag{2.7}$$

In analogy with the Andronov–Hopf bifurcation in two-dimensional space this value characterize energetic relations in the system (1.8).

Calculating the derivative of value (2.7) due to the system (1.8) one can easily find that:

$$\dot{u} = 2\,\mu\,u - 4\,u \star u\,. \tag{2.8}$$

After rescaling of time $t \mapsto 2\,\mu\,t$ and value (2.7) $u \mapsto 2\,\dfrac{u}{\mu}$ the Cauchy problem for equation (2.8) is reduced to the Cauchy problem (2.1) with initial condition $u^0 = \dfrac{v^0 \star v^0 + w^0 \star w^0}{\mu}$, where $v^0 \in \mathbb{L}$ and $w^0 \in \mathbb{L}$ are initial conditions for the system (1.8).

Thus, the Cauchy problem (2.1) (such as the problem (2.2)) meets quite often in different applications.

For the Cauchy problem (2.2) the following theorem is valid.

Theorem 2.1. *General representation of exact solution of the Cauchy problem* (2.2) *is equal to:*

$$u_n(t) = \frac{1}{2\,\pi\,i}\oint\limits_{C_\rho} \frac{U^0(z)}{U^0(z) + \exp(-t)\,[\,1 - U^0(z)]}\,\frac{dz}{z^{n+1}}\,, \tag{2.9}$$

where

$$U^0(z) = \sum_{n=-\infty}^{n=+\infty} u_n^0\,z^n, \quad z \in K \tag{2.10}$$

and K is the ring of analyticity of the integrand in formula (2.9), *integration along the circle $C_\rho = \{z \in \mathbb{C} \mid |\,z\,| = \rho\} \in K$ being counter clockwise.*

Доказательство.

Let us introduce for solution $u(t) \in \mathbb{L}$ of equation (2.1) the generating function:

$$U(z,t) = \sum_{n=-\infty}^{n=+\infty} u_n(t)\,z^n, \quad z \in K\,. \tag{2.11}$$

We stress that unlike the generating function (1.5) the generating function (2.11) depends both complex variable z and real variable t.

Applying formula (1.6) to the system (2.2) it is not difficult to observe that the input Cauchy problem (2.2) is reduced to the next one:

$$\frac{\partial U(z,t)}{\partial t} = U(z,t) - U^2(z,t), \quad U(z,0) = U^0(z), \tag{2.12}$$

where initial condition for equation (2.12) is equal to the generating function (2.10) for vector $u^0 \in \mathbb{L}$:

Further it is well-known that exact solution of the Cauchy problem (2.12) is equal to:

$$U(z,t) = \frac{U^0(z)}{U^0(z) + \exp(-t)\,[\,1 - U^0(z)]}\,. \tag{2.13}$$

Combining formulas (2.11) and (2.13) in the framework of the theory of analytical functions one can obtain expression (2.9).
Доказательство завершено.

Remark 2.1. *Singular points of expression* (2.13) *may depend on time t hence internal and external radii of the analyticity ring K for function* (2.13) *may depend on time t too: $r = r(t)$ and $R = R(t)$. Therefore radius ρ of the circle of integration $C_\rho \subset K$ in expression* (2.9) *generally speaking also depends on time t: $\rho = \rho(t)$.*

3. Examples of exact solutions of countable-dimensional logistic model

In practice instead of application of expression (2.9) it is more suitable to derive the Laurent expansion (2.11) and hence to extract exact solution $u(t) \in \mathbb{L}$ of the input Cauchy problem (2.1) directly from formula (2.13) using initial condition (2.10).

The next statement demonstrate this procedure.

Theorem 3.1. *Exact solution of the Cauchy problem* (2.2) *under the following nonzero values for initial condition $u^0 \in \mathbb{L}$:*

$$u_0^0 = a\,, \quad u_1^0 = u_{-1}^0 = \frac{b}{2}\,, \quad 0 < a - b < a + b < 1\,, \tag{3.1}$$

is equal to

$$u_0(t) = \frac{1}{\sqrt{A^2(t) - B^2(t)}}\left[a - \frac{b\,B(t)}{A(t) + \sqrt{A^2(t) - B^2(t)}}\right], \tag{3.2}$$

and

$$u_n(t) = u_{-n}(t) = (-1)^{n+1}\,\frac{b\,\exp(-t)}{\sqrt{A^2(t) - B^2(t)}}\,\frac{B^{n-1}(t)}{(A(t) + \sqrt{A^2(t) - B^2(t)})^n}, \quad n \in \mathbb{N}\,, \tag{3.3}$$

where $A(t) = a + (1-a)\exp(-t)$ and $B(t) = b\,(1 - \exp(-t))$.

P r o o f. First of all let one restore on values (3.1) corresponding to them initial generating function (2.10):

$$U^0(z) = a + \frac{b}{2}\left(z + \frac{1}{z}\right), \qquad z \in K. \tag{3.4}$$

The result of substitution of formula (3.4) into expression (2.13) is equal to

$$U(z,t) = \frac{2\,a\,z + b\,(z^2+1)}{2\,A(t)\,z + B(t)\,(z^2+1)}\,. \tag{3.5}$$

Denominator of formula (3.5) has two zeros, namely, $z_1 = \zeta(t)$ and $z_2 = \dfrac{1}{\zeta(t)}$, where

$$\zeta(t) = -\frac{B(t)}{A(t) + \sqrt{A^2(t) - B^2(t)}}\,, \tag{3.6}$$

hence expansion of function (3.5) into the Laurent series on the open ring K_t is equal to

$$U(z,t) = u_0(t) + u_1(t)\sum_{n=1}^{+\infty}\zeta^{n-1}(t)\,(z^n + z^{-n}), \quad z \in K_t\,, \tag{3.7}$$

where $K_t = \{z \in \mathbb{C} \mid |\zeta(t)| < |z| < |\zeta(t)|^{-1}\}$,

$$u_0(t) = \frac{a + b\,\zeta(t)}{\sqrt{A^2(t) - B^2(t)}} \tag{3.8}$$

and

$$u_1(t) = \frac{b\,\zeta^2(t) + 2\,a\,\zeta(t) + b}{2\,\sqrt{A^2(t) - B^2(t)}}\,. \tag{3.9}$$

Thus from formula (3.7) it is obvious that exact solution of the Cauchy problem (2.2) under initial condition (3.1) is equal to

$$u_n(t) = u_{-n}(t) = u_1(t)\,\zeta^{n-1}(t), \quad n \in \mathbb{N}\,. \tag{3.10}$$

Inserting expression (3.6) into formulas (3.8), (3.9) and (3.10), one can easily obtain formulas (3.2) and (3.3) in the statement of the theorem.

P r o o f f i n i s h e d.

Temporal evolution of the first functions (3.8), (3.9) and (3.10) assuming $a = 0.5$ and $b = 0.25$ is shown on Figure 3.1.

From formulas (3.2) and (3.3) it is easy to find that:

$$\lim_{t\to+\infty} u_0(t) = 1, \quad \lim_{t\to+\infty} u_n(t) = 0, \quad n \in \mathbb{N}\,. \tag{3.11}$$

It means that $\lim\limits_{t\to+\infty} u(t) = e$, where e is the unit of commutative algebra $\mathbb{L}$. On the other side the result (3.11) can be reformulated on the language of generating function (3.5), namely, $\lim\limits_{t\to+\infty} U(z,t) = 1$. Both languages are useful.

Further in correspondence with Remark 2.1 in this case the ring of analyticity K_t for generating function (3.5) evolves in time, namely, its internal $r(t)$ and external $R(t)$ radii depend on t as follows: $r(t) = |\zeta(t)|$ and $R(t) = \dfrac{1}{|\zeta(t)|}$. Typical graphs of these functions are presented on Figure 3.2.

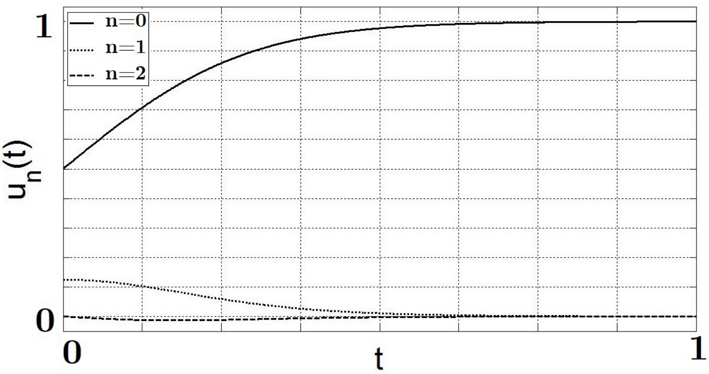

Рис. 3.1. Зависимость от времени первых трёх функций $u_n(t)$ при $a = 0.5$ и $b = 0.25$

Fig. 3.1. Temporal evolution of the first functions $u_n(t)$ under $a = 0.5$ and $b = 0.25$

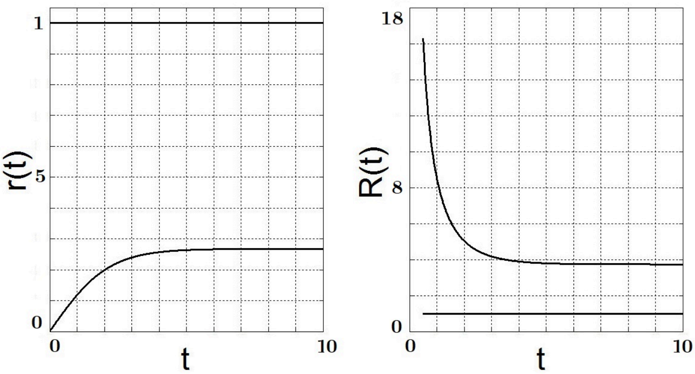

Рис. 3.2. Зависимость от времени внутреннего (справа) и внешнего (слева) радиусов кольца аналитичности производящей функции $U(z,t)$ при $a = 0.5$ и $b = 0.25$

Fig. 3.2. Temporal evolution of internal (on the left) and external (on the right) radii of the ring of analyticity of generating function $U(z,t)$ under $a = 0.5$ and $b = 0.25$

From this Figure 3.1 one can see that graph of $r(t)$ is monotone increasing and graph of $R(t)$ is monotone decreasing. Moreover using expression (3.6) it is not difficult to obtain that:

$$\lim_{t\to+\infty} r(t) = \frac{b}{a+\sqrt{a^2-b^2}} < 1\,, \qquad \lim_{t\to+\infty} R(t) = \frac{a+\sqrt{a^2-b^2}}{b} > 1\,. \tag{3.12}$$

It means graphs $r(t)$ and $R(t)$ possesses by horizontal asymptotes (see Figure 3.2). And inequalities (3.12) claim that there is a gap between this asymptotes therefore under $t \to +\infty$ there is a limit ring $K_\infty = \left\{ z \in \mathbb{C} \mid \dfrac{b}{a+\sqrt{a^2-b^2}} < |z| < (a + \dfrac{\sqrt{a^2-b^2}}{b} \right\}$ on which the generating function (3.5) is always analytical. Thus one can consider this generating function only on K_∞ at once. Further due to inequalities (3.12) the unit circle $C_1 \subset K_\infty$ hence in formula (2.9) it is possible to integrate along the unit circle $|z| = 1$ always because of deformation of the contour of integration.

In conclusion of this section let us perform the following observation, namely, let on change in the Laurent expansion (3.7) complex variable z on some its degree z^p, where $p = 2, 3, 4, \dots$, then:

$$\tilde{U}(z,t) \equiv U(z^p,t) = u_0(t) + u_1(t) \sum_{n=1}^{+\infty} \zeta^{n-1}(t)\,(z^{n\,p} + z^{-n\,p})\,. \tag{3.13}$$

On the other hand equation (2.12) do not contain variable z explicitly therefore expression (3.13) represents exact solution of equation (2.12) too. But this solution corresponds to the next initial condition:

$$\tilde{U}^0(z) \equiv U^0(z^p) = a + \frac{b}{2}\left(z^p + \frac{1}{z^p}\right). \tag{3.14}$$

In other words expansion (3.13) gives one exact solution of the Cauchy problem (2.1) as follows:

$$\tilde{u}_0(t) = u_0(t)\,, \qquad \tilde{u}_{\pm np}(t) = u_n(t)\,, \qquad n \in \mathbb{N}\,, \tag{3.15}$$

where $u_n(t)$ are functions (3.2) and (3.3), and place between components of vector $\tilde{u}(t) \in \mathbb{L}$ with numbers $n\,p$ and $n\,p + p$ both in initial condition (3.14) and exact solution (3.13) are filled by zeros.

From point of view of radio engineering procedure (3.15) just coincides with procedure of upsampling in digital signal processing [12].

This observation one may formulate as the following statement.

Theorem 3.2. *If generating function $U(z,t)$ represents exact solution of the Cauchy problem* (2.1) *with initial condition representing by generating function $U^0(z)$, then for all $p = 2, 3, 4, \dots$ generating function $U(z^p,t)$ represents exact solution of the Cauchy problem* (2.1) *with initial condition representing by generating function $U^0(z^p)$.*

4. General solution of countable-dimensional system of nonlinear partial differential equations

Let one consider the following Cauchy problem for vector $u(x,t) \in \mathbb{L}$:

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} + u \star u = 0\,, \quad u(x,0) = u^0(x) \in \mathbb{L}\,, \quad x \in \mathbb{R}\,, \quad c \in \mathbb{R}\,. \tag{4.1}$$

This Cauchy problem may be interpreted as nonlinear transfer of initial condition $u^0(x)$ in the space $\mathbb{L}$.

Using formula (1.7) one can rewrite equation (4.1) in components as countable-dimensional system of partial differential equations:

$$\frac{\partial u_n(x,t)}{\partial t} + c\,\frac{\partial u_n(x,t)}{\partial x} + \sum_{k=-\infty}^{+\infty} u_{n-k}(x,t)\,u_k(x,t) = 0, \quad u_n(x,0) = u_n^0(x), \quad n \in \mathbb{Z}. \tag{4.2}$$

It is easy to see that system (4.2) arises from the Cauchy problem for the following nonlinear integro-differential equation:

$$\frac{\partial \hat{U}(x,y,t)}{\partial t} + c\,\frac{\partial \hat{U}(x,y,t)}{\partial x} + \int\limits_{-\infty}^{+\infty} \hat{U}(x,y-\eta,t)\,\hat{U}(x,\eta,t)\,d\eta = 0\,. \tag{4.3}$$

namely, quantization of plain $\mathbb{R}^2$ in y-direction with step δ generates from function $u(x,y,t)$ countable set of functions $u_n(x,t) = \hat{U}(x, n\,\delta, t)$, $n \in \mathbb{Z}$. And after that change of integral in equation (4.3) by corresponding to it integral sum and further scaling $\delta\, u_n \to u_n$ ought to give rise to system (4.2) exactly.

Exact solution of the Cauchy problem (4.1) (such as (4.2)) can be obtained by means of the following theorem.

Theorem 4.1. *General representation of exact solution of the Cauchy problem* (4.2) *is equal to:*

$$u_n(x,t) = \frac{1}{2\,\pi\,i}\oint\limits_{C_\rho} \frac{U^0(z; x - c\,t)}{1 + t\,U^0(z; x - c\,t)}\,\frac{dz}{z^{n+1}}\,, \tag{4.4}$$

where

$$U^0(z;x) = \sum_{n=-\infty}^{+\infty} u_n^0(x)\,z^n, \quad z \in K\,, \tag{4.5}$$

and K is the ring of analyticity of the integrand in formula (4.4), *integration along the circle $C_\rho = \{z \in \mathbb{C} \mid |z| = \rho\} \in K$ being counter clockwise.*

Proof. Let us introduce the generating function for the solution $u(x,t) \in \mathbb{L}$ of the Cauchy problem (4.2):

$$U(z;x,t) = \sum_{n=-\infty}^{+\infty} u_n(x,t)\,z^n\,. \tag{4.6}$$

Then, using the Laurent expansion (4.6) and formula (1.6) one can rewrite system (4.2) as follows:

$$\frac{\partial U(z;x,t)}{\partial t} + c\,\frac{\partial U(z;x,t)}{\partial x} + U^2(z;x,t) = 0, \quad U(z;x,0) = U^0(z;x)\,. \tag{4.7}$$

Equation (4.7) is the first order partial differential equation, therefore, the Cauchy problem (4.7) can be easily solved in the framework of the well-known method of characteristics:

$$U(z;x,t) = \frac{U^0(z; x - c\,t)}{1 + t\,U^0(z; x - c\,t)}. \tag{4.8}$$

Expression (4.8) describes the nonlinear transfer of the initial condition (4.5) along the x-axis at a speed c.

Comparing expressions (4.6) and (4.8) and using the theory of analytical functions it is not difficult to find general representation (4.4) of exact solution of the Cauchy problem (4.2) (such as (4.1)).

Proof finished.

Remark 4.1. *Singular points of expression* (4.8) *may depend on time t and coordinate x hence internal and external radii of the analyticity ring K for function* (4.8) *may depend on time t and coordinate x too:* $r = r(x,t)$ *and* $R = R(x,t)$. *Therefore radius* ρ *of the integration circle* $C_\rho \subset K$ *in expression* (4.4) *generally speaking also depends on time t and coordinate x:* $\rho = \rho(x,t)$.

5. Examples of exact solutions of countable-dimensional system of nonlinear partial differential equations

In practice instead of usage of formula (4.4) it is more convenient to derive the Laurent expansion (4.6) and to extract exact solution $u(x,t) \in \mathbb{L}$ of the Cauchy problem (4.1) from formula (4.8) straightforwardly.

The next theorem illustrates this approach.

Theorem 5.1. *Exact solution of the Cauchy problem* (4.2) *under the following nonzero values for initial condition* $u^0(x) \in \mathbb{L}$*:*

$$u_0^0(x) = a(x), \quad u_1^0(x) = u_{-1}^0(x) = \frac{b(x)}{2}, \quad x \in \mathbb{R}, \tag{5.1}$$

is equal to

$$u_0(x,t) = \frac{a(x-c\,t) + b(x-c\,t)\,\zeta(x,t)}{\sqrt{A^2(x,t) - B^2(x,t)}}, \tag{5.2}$$

$$u_1(x,t) = \frac{b(x-c\,t)\,\zeta^2(x,t) + 2\,a(x-c\,t)\,\zeta(x,t) + b(x-c\,t)}{2\,\sqrt{A^2(x,t) - B^2(x,t)}}, \tag{5.3}$$

$$u_n(x,t) = u_{-n}(x,t) = u_1(x,t)\,\zeta^{n-1}(x,t), \quad n \in \mathbb{N}. \tag{5.4}$$

$$\zeta(x,t) = -\frac{B(x,t)}{A(x,t) + \sqrt{A^2(x,t) - B^2(x,t)}}, \tag{5.5}$$

and $A(x,t) = 1 + t\,a(x-c\,t)$, $B(x,t) = t\,b(x-c\,t)$.

Proof.

First of all let us construct from initial conditions (5.1) initial generating function (4.5):

$$U^0(z;x) = a(x) + \frac{b(x)}{2}\left(z + \frac{1}{z}\right). \tag{5.6}$$

After that substituting function (5.6) in formula (4.8) one can obtain:

$$U(z;x,t) = \frac{2\,a(x-c\,t)\,z + b(x-c\,t)\,(z^2+1)}{2\,A(x,t)\,z + B(x,t)\,(z^2+1)}. \tag{5.7}$$

The structure of expression (5.7) coincides with the structure of expression (3.5). In particular, the denominator of the expression (5.7) has two zeros: $z_1 = \zeta(x,t)$ and $z_2 = \dfrac{1}{\zeta(x,t)}$. Thus, in full analogy with formula (3.7), the Laurent series expansion of function (5.7) is:

$$U(z;x,t) = u_0(x,t) + u_1(x,t)\sum_{n=1}^{+\infty}\zeta^{n-1}(x,t)\,(z^n + z^{-n}). \tag{5.8}$$

The series (5.8) converges in the variable z in an open ring:

$$K_{x,t} = \left\{ z \in \mathbb{C} \mid \; |\zeta(x,t)| < |z| < \left| \frac{1}{\zeta(x,t)} \right| \right\} . \tag{5.9}$$

P r o o f f i n i s h e d.

R e m a r k 5.1. *The exact solution* (5.2)–(5.5) *of the Cauchy problem* (4.2) *is given by two arbitrary functions $a(x)$ and $b(x)$. It is obvious that these functions must be differentiable on the entire number line. In addition, at each point (x,t), these functions must allow the existence of the open ring* (5.9)*.*

To illustrate the application of Theorem 5.1, let us choose the following initial conditions:

$$a(x) = \frac{1}{2} , \qquad b(x) = \frac{1}{4} + \frac{1}{8} \cos x , \tag{5.10}$$

and let the transfer rate be $c = 0.2$.

The functions (5.10) are periodic with a period 2π, so in this case the exact solution (5.2)–(5.5) can only be considered for $x \in [0, 2\pi]$.

Let $t \in [0, 10]$. Figure 5.1 shows the graph of the auxiliary function (5.5), which corresponds to the functions (5.10), on the open rectangle $\Pi = (0, 2\pi) \times (0, 10)$.

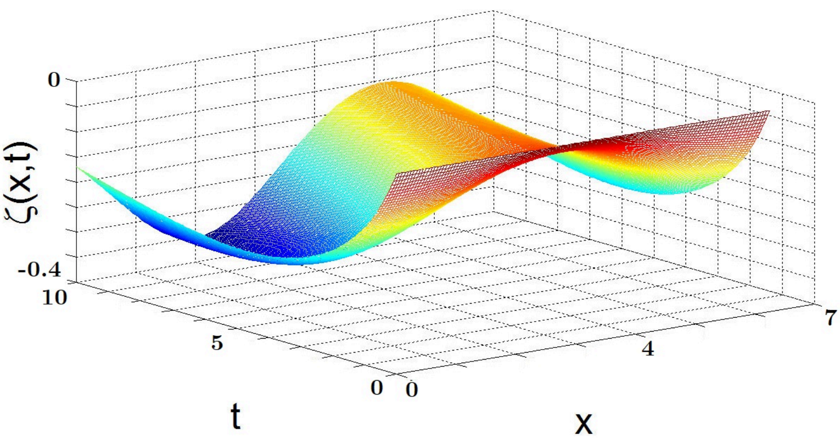

Рис. 5.1. Пространственно-временная эволюция вспомогательной функции $\zeta(x,t)$

Fig. 5.1. Spatiotemporal evolution of auxiliary function $\zeta(x,t)$

It can be seen from Figure 5.1 that $\sup_\Pi |\zeta(x,t)| < 1$, therefore, $\inf_\Pi \left| \dfrac{1}{\zeta(x,t)} \right| > 1$. Further, let $K_\Pi = \left\{ z \in \mathbb{C} \mid \sup_\Pi |\zeta(x,t)| < |z| < \inf_\Pi \left| \dfrac{1}{\zeta(x,t)} \right| \right\}$. For all $(x,t) \in \Pi$ this open ring lies inside the open ring (5.9): $K_\Pi \subseteq K_{x,t}$. It means that the Laurent series (5.8) converges in the open ring K_Π, and for the integration contour C_ρ in formula (4.4) we can set $\rho = 1$.

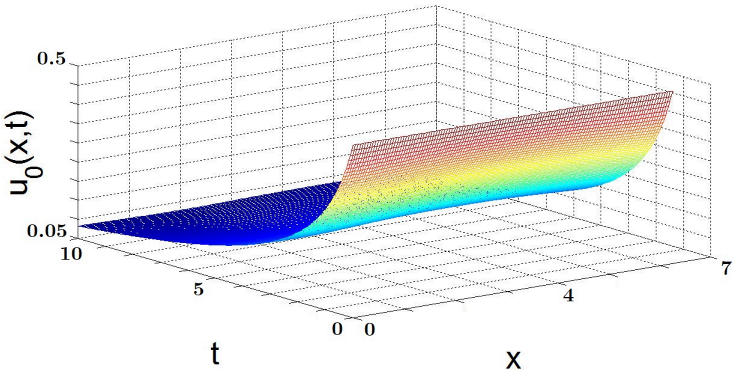

Рис. 5.2. Пространственно-временная эволюция функции $u_0(x,t)$

Fig. 5.2. Spatiotemporal evolution of function $u_0(x,t)$

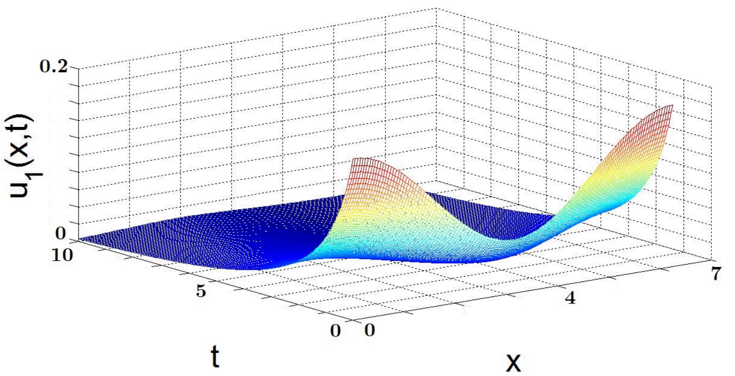

Рис. 5.3. Пространственно-временная эволюция функции $u_1(x,t)$

Fig. 5.3. Spatiotemporal evolution of function $u_1(x,t)$

The graphs of the first four functions $u_n(x,t)$ ($n = \overline{0,3}$) on the rectangle Π are presented in Figures 5.2-5.5.

In conclusion of this section let one observe that equation (4.7) do not contain variable z explicitly. This means that the following statement is true, which is quite similar to Theorem 3.2, namely:

Theorem 5.2. *If generating function $U(z;x,t)$ represents exact solution of the Cauchy problem* (4.1) *with initial condition representing by generating function $U^0(z;x)$, then for all $p = 2,3,4,\ldots$ generating function $U(z^p;x,t)$ represents exact solution of the Cauchy problem* (4.1) *with initial condition representing by generating function $U^0(z^p;x)$.*

To illustrate Theorem 5.2, consider the following Laurent series expansion, constructed using the functions (5.2), (5.3) and (5.5):

$$\tilde{U}(z;x,t) \equiv U(z^p;x,t) = u_0(x,t) + u_1(x,t) \sum_{n=1}^{+\infty} \zeta^{n-1}(x,t)\,(z^{n\,p} + z^{-n\,p})\,. \qquad (5.11)$$

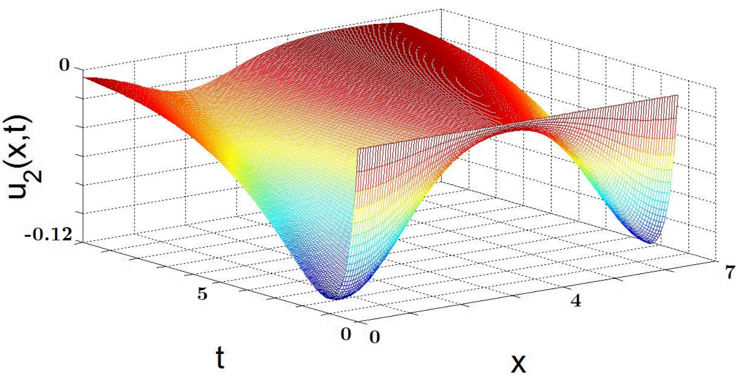

Рис. 5.4. Пространственно-временная эволюция функции $u_2(x,t)$

Fig. 5.4. Spatiotemporal evolution of function $u_2(x,t)$

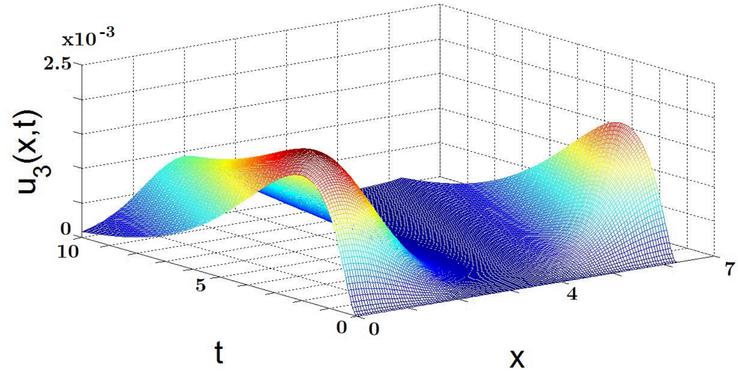

Рис. 5.5. Пространственно-временная эволюция функции $u_3(x,t)$

Fig. 5.5. Spatiotemporal evolution of function $u_3(x,t)$

According to Theorem 5.2, the function (5.11) is an exact solution of the equation (4.7), with the initial condition given by the function:

$$\tilde{U}^0(z;x) \equiv U^0(z^p;x) = a(x) + \frac{b(x)}{2}\left(z^p + \frac{1}{z^p}\right). \qquad (5.12)$$

Thus, expansion (5.11) gives one exact solution of the countable-dimensional system of partial differential equations (4.2) with initial condition (5.12) as follows:

$$\tilde{u}_0(x,t) = u_0(x,t)\,, \qquad \tilde{u}_{\pm np}(x,t) = u_n(x,t)\,, \qquad n \in \mathbb{N}\,, \qquad (5.13)$$

where $u_n(x,t)$ are functions (5.2)–(5.4), and place between components of vector $\tilde{u}(x,t) \in \mathbb{L}$ with numbers $n\,p$ and $n\,p+p$ both in initial condition (5.12) and exact solution (5.11) are filled by zeros.

Structure of exact solution (5.13) is completely analogous to structure of signal under procedure of unsumpling in digital signal processing [12].

6. Conclusion

The work deals with dynamical systems in the linear space $\mathbb{L}$ of double-sided sequences. This countable-dimensional space is described by Definition 1.2.

In the article general solution (2.9) of the Cauchy problem (2.1) in this space has been done. This dynamical system arises from the spatial discretization of nonlinear integro-differential equation (2.3). Theorem 3.1 provides specific example of exact solution to Cauchy problem (2.1), and Theorem 3.2 demonstrates how this exact solution generates a countable set of exact solutions for the dynamical system (2.1). These specific examples of exact solutions are important for illustrating the application of general theorems in the theory of countable-dimensional systems of ordinary differential equations, which was introduced in article [13]. In addition, this result extends and complements our knowledge of the integrability of nonlinear countable-dimensional systems of ordinary differential equations, which dates back to works [14–15].

General solution (4.4) of the Cauchy problem (4.1) extends above-described approaches on the case of distributed dynamical system in the space of double-sided sequences.

Of significant interest is the extension of the approaches developed in this article to both analogues of discrete dynamical systems in the space $\mathbb{L}$ and countable-dimensional analogues of systems of ordinary differential equations with the property of deterministic chaos [16–18].

Acknowledgements. The publication was prepared within the framework of the Academic Fund Program at HSE University (grant No. 25-00-014 "A new numerical-analytical method for solving Cauchy problems for evolutionary equations and systems of evolutionary equations with quadratic nonlinearity").

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Submitted 10.11.2021; Revised 19.02.2022; Accepted 24.02.2022

The author has read and approved the final manuscript.
Conflict of interest: The author declare no conflict of interest.

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Поступила 10.11.2021; доработана после рецензирования 19.02.2022; принята к публикации 24.02.2022

Авторы прочитали и одобрили окончательный вариант рукописи.
Конфликт интересов: авторы заявляют об отсутствии конфликта интересов.