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A Study of Numerical Methods for Solving the Nonlinear Energy Resources Supply-Demand System

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Abstract. In this study, we implement and estimate various numerical methods for solving a nonlinear differential equation system modeling energy resources supply-demand dynamics. Both single-step methods (Taylor series, Runge-Kutta) and multi-step methods (Adams – Bashforth, Adams Predictor-Corrector) are employed. In addition to standard fourth-order methods, higher-order techniques such as the fifth-order Runge-Kutta method and the sixth-order Taylor series method are also applied. Furthermore, along with fixed-step numerical methods, we implement and assess adaptive step-size methods, including the explicit Runge-Kutta method of order 5(4) (that is RK45), the explicit Runge-Kutta method of order 8(5,3) (or DOP853), the implicit Runge-Kutta method from the Radau IIA family of order 5 (Radau), the implicit method based on backward differentiation formulas (BDF), and the Adams/BDF method with automatic switching (LSODA). The results indicate that, in the cases we considered, single-step methods are more effective than multi-step ones in capturing and tracking rapid variations of the system, while multi-step methods require less computation time. Adaptive step-size numerical methods demonstrate both flexibility and stability. Through the evaluation and analysis of numerical solutions obtained by various methods, the behaviour and dynamic characteristics of the system are explored.

Keywords: energy supply and demand system, Runge-Kutta method, Taylor series, Adams-Bashforth method, Adams Predictor-Corrector method, RK45, DOP853, Radau, BDF, LSODA

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Оригинальная статья

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Исследование численных методов решения нелинейной системы спроса и предложения энергетических ресурсов

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Аннотация. В данном исследовании реализованы и оценены различные численные методы для решения нелинейной системы дифференциальных уравнений, моделирующей динамику спроса и предложения энергетических ресурсов. Использованы как одношаговые методы (ряд Тейлора, метод Рунге–Кутты), так и многошаговые методы (Адамса–Башфорта, метод прогноза–коррекции Адамса). Помимо стандартных методов четвёртого порядка, применялись также методы более высокого порядка, такие как метод Рунге–Кутты пятого порядка и метод ряда Тейлора шестого порядка. Кроме того, наряду с численными методами с фиксированным шагом, были реализованы и оценены методы с адаптивным шагом, включая явный метод Рунге–Кутты порядка 5(4) (RK45), явный метод Рунге–Кутты порядка 8(5,3) (DOP853), неявный метод Рунге–Кутты семейства Radau IIA порядка 5 (Radau), неявный метод на основе формул обратного дифференцирования (BDF), а также комбинированный метод Адамса/BDF с автоматическим переключением (LSODA). Полученные результаты показывают, что в рассмотренных случаях одношаговые методы были более эффективны, чем многошаговые, при отслеживании быстрых изменений системы, тогда как многошаговые методы требовали меньше времени на вычисления. Численные методы с адаптивным шагом продемонстрировали как гибкость, так и устойчивость. Посредством оценки и анализа численных решений, полученных различными методами, исследуются динамические характеристики и поведение системы.

Ключевые слова: система спроса и предложения энергии, метод Рунге–Кутты, ряд Тейлора, метод Адамса–Башфорта, метод прогнозирования–коррекции Адамса, RK45, DOP853, Radau, BDF, LSODA

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1. Introduction

Numerical methods constitute a major branch of mathematical research, aimed at solving problems where finding explicit analytical solutions is challenging. One such class of problems includes nonlinear differential equations and their systems. In mathematics, differential equations and their systems, particularly initial value problems, have been extensively studied due to their role as powerful tools for modeling real-world problems [1–2]. In the context of the growing urgency to use energy resources efficiently and sustainably, forecasting and stabilizing energy resources supply and demand play a crucial role. Mei Sun et al. proposed a nonlinear system of differential equations to describe the energy resources supply-demand system [3–4], which was formulated based on a real-world problem to ensure the stability of energy resources supply and demand between the eastern and western regions of China. Studies have shown that this system exhibits chaotic and highly nonlinear behavior [5], making it extremely complex to obtain exact solutions. In this case, employing numerical methods to obtain approximate solutions presents a more practical and feasible approach.

Numerical methods constitute a broad research field encompassing various techniques that have been extensively developed and studied, each demonstrating suitability and effectiveness for specific types of problems [6]. However, for the nonlinear energy resources supply-demand system, most existing studies primarily focus on stabilizing the chaotic behavior of the system [7–8], while in-depth investigations into numerical methods for solving the system remain limited. Recently, a study by Vo et al. proposed using physics-informed neural networks to solve this system. Although this approach shows great potential, it has certain drawbacks, such as requiring significant computational power from modern computing systems and being time-consuming due to the need for model training [9].

In this paper, we conduct an in-depth study by implementing numerical methods to solve the nonlinear energy resources supply-demand system. The numerical methods employed include representative one-step methods such as the Taylor series method and the Runge-Kutta method. Multi-step methods such as Adams-Bashforth, Adams-Moulton, and the Adams-Predictor-Corrector method [10] are also considered. Additionally, adaptive step-size methods such as: RK45 [11–12], DOP853 [13], Radau [14], BDF [15], LSODA [16] are implemented. Notably, for fixed-step methods, we deploy higher-order approaches such as RK5 and the sixth-order Taylor series method. The numerical solutions obtained from these higher-order methods are then used to assess the accuracy of lower-order fourth-order methods. Beyond solving the system, this study also evaluates and compares numerical methods based on criteria such as accuracy, convergence speed, computational efficiency, and stability, aiming to identify the most suitable and effective methods for the given system. All numerical methods are implemented in Python [17], For adaptive step-size methods, we

utilize numerical solvers from the SciPy library [18], which is integrated into the Python programming language.

Main contributions of the Study: Implement various numerical methods for solving the nonlinear energy resources supply-demand system. Implement high-order numerical methods to solve the nonlinear energy resources supply-demand system. Analyze, evaluate, and compare the effectiveness of numerical methods for the nonlinear differential equation system describing the energy resources supply-demand problem. Analyze and explore the behavioral characteristics of the system through numerical solutions.

The paper is structured into five main sections: Section 1. provides a general introduction. Section 2. describes the nonlinear differential equation system for the energy resources supply-demand problem. Section 3. presents the numerical methods used in this study to solve the energy resources supply-demand system. Section 4. proposes several approaches for analyzing and evaluating the effectiveness of the implemented numerical methods. Finally, Section 5. presents the main results of the study.

2. System description

To model the distribution of energy resources supply and demand between different regions C and D, Mei et al. formulated three-dimensional and four-dimensional nonlinear differential equation systems. In this study, we focus on numerically solving the four-dimensional energy resources supply-demand system. The mathematical formulation of this system is given below [4]:

$$\begin{cases} x_1'(t) = a_1 x_1(t)(1 - \frac{x_1(t)}{M}) - a_2(x_2(t) + x_3(t)) - d_3 x_4(t), \\ x_2'(t) = -z_1 x_2(t) - z_2 x_3(t) + z_3 x_1(t)[N - (x_1(t) - x_3(t))], \\ x_3'(t) = s_1 x_3(t)(s_2 x_1(t) - s_3), \\ x_4'(t) = d_1 x_1(t) - d_2 x_4(t), \end{cases} \tag{2.1}$$

The initial conditions of the system (2.1) are given by $x_i(t_0) = b_i$, and [9] $x_i'(t)$ denotes the derivative of the function $x_i(t)$ with respect to the variable t, where $i \in \{1, 2, 3, 4\}$. $x_i(t)$ are the unknown functions to be determined, where the energy demand in region D is represented by $x_1(t)$, the energy supply from region C to region D is represented by $x_2(t)$, the imported energy resources into region D are represented by $x_3(t)$, and the renewable energy resources in region D are represented by $x_4(t)$. With the constants of the system satisfying the following conditions $a_j, d_j, z_j, s_j, N, M > 0$ and $N < M$. The system (2.1) is in a chaotic state with the coefficients $a_1 = 0.09$, $a_2 = 0.15$, $z_1 = 0.06$, $z_2 = 0.082$, $z_3 = 0.07$, $s_1 = 0.2$, $s_2 = 0.5$, $s_3 = 0.4$, $M = 1.8$, $N = 1$, $d_1 = 0.1$, $d_2 = 0.06$, $d_3 = 0.08$, and the initial conditions $x_1(0) = 0.82$, $x_2(0) = 0.29$, $x_3(0) = 0.48$, $x_4(0) = 0.1$ [4–5].

3. Numerical Methods

To facilitate the presentation of numerical methods, the differential equation system (2.1) is rewritten as follows:

Let f_1, f_2, f_3, f_4 denote the functions representing the right-hand side of the differential equation system (2.1), and let b_1, b_2, b_3, b_4 denote the initial values corresponding to the

four components of the system, given by:

$$f_1(t, x_1, x_2, x_3, x_4) = a_1 x_1(t)\left(1 - \frac{x_1(t)}{M}\right) - a_2(x_2(t) + x_3(t)) - d_3 x_4(t),$$

$$f_2(t, x_1, x_2, x_3, x_4) = -z_1 x_2(t) - z_2 x_3(t) + z_3 x_1(t)[N - (x_1(t) - x_3(t))],$$

$$f_3(t, x_1, x_2, x_3, x_4) = s_1 x_3(t)(s_2 x_1(t) - s_3),$$

$$f_4(t, x_1, x_2, x_3, x_4) = d_1 x_1(t) - d_2 x_4(t),$$

we can rewrite the general form of the system as follows:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \ , \ \mathbf{x}(t_0) = \mathbf{b}.$$

where $\mathbf{x}$, $\mathbf{f}$, and $\mathbf{b}$ are vectors defined as follows:

$$\mathbf{x} = [x_1, x_2, x_3, x_4]^T, \ \mathbf{f} = [f_1, f_2, f_3, f_4]^T, \ \mathbf{b} = [b_1, b_2, b_3, b_4]^T.$$

In this study, we define $\mathbf{x}(t_i)$ as the exact solutions of the system at time t_i and denote $\mathbf{x}_i$ as the approximate solutions of the system at time t_i, obtained using numerical methods.

3.1. Single step methods

In the field of numerical methods, a class of techniques for solving differential equation systems in which the solution at any time t_{i+1} is approximated solely based on the solution at the previous time step t_i is referred to as single-step numerical methods [10].

3.1.1. Taylor Series Method

Taylor's method is a powerful explicit single-step numerical method [19–20], Its foundation is based on the Taylor series expansion, a fundamental concept in function approximation theory. According to this principle, the value of a function at any given point can be approximated using a nearby known point, represented by a polynomial and its higher-order derivatives, as given by the Taylor series. In the Taylor numerical method, the solution at time t_{i+1} is approximated using its value at t_i through the following formula [10]:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h\mathbf{x_i}' + \frac{h^2}{2!}\mathbf{x_i}'' + \frac{h^3}{3!}\mathbf{x_i}''' + ... + \frac{h^p}{p!}\mathbf{x_i}^{(p)}, \tag{3.1}$$

where:

h: is the step size.

p: is the order of the derivative.

$\mathbf{x}_i^{(k)}$: the values of the k-th order derivatives of the solution functions at time t_i.

To implement this numerical method, one of the crucial aspects is determining the expressions for the higher-order derivatives of the system. This task is not straightforward. In this study, we have determined the expressions for computing the derivatives up to the sixth order for the solution functions of the differential equation system (2.1). The detailed expressions are presented below:

Expressions for the first-order derivatives of the solution functions:

$$x_1^{(1)}(t) = a_1 x_1(t)(1 - \frac{x_1(t)}{M}) - a_2(x_2(t) + x_3(t)) - d_3 x_4(t),$$

$$x_2^{(1)}(t) = -z_1x_2(t) - z_2x_3(t) + z_3x_1(t)[N - (x_1(t) - x_3(t))],$$
$$x_3^{(1)}(t) = s_1x_3(t)(s_2x_1(t) - s_3),$$
$$x_4^{(1)}(t) = d_1x_1(t) - d_2x_4(t).$$

Expressions for the second-order derivatives of the solution functions:

$$x_1^{(2)}(t) = a_1x_1^{(1)}(t)\left[1 - \frac{2x_1(t)}{M}\right] - a_2\left[x_2^{(1)}(t) + x_3^{(1)}(t)\right] - d_3x_4^{(1)}(t),$$
$$x_2^{(2)}(t) = -z_1x_2^{(1)}(t) - z_2x_3^{(1)}(t) + z_3x_1(t)\left[x_3^{(1)}(t) - x_1^{(1)}(t)\right] + z_3x_1^{(1)}(t)\left[N - x_1(t) + x_3(t)\right],$$
$$x_3^{(2)}(t) = s_1s_2x_3(t)x_1^{(1)}(t) + s_1x_3^{(1)}(t)\left[s_2x_1(t) - s_3\right],$$
$$x_4^{(2)}(t) = d_1x_1^{(1)}(t) - d_2x_4^{(1)}(t).$$

Expressions for the third-order derivatives of the solution functions:

$$x_1^{(3)}(t) = a_1x_1^{(2)}(t)\left[1 - \frac{2x_1(t)}{M}\right] - a_2\left[x_2^{(2)}(t) + x_3^{(2)}(t)\right] - d_3x_4^{(2)}(t) - 2a_1\frac{\left[x_1^{(1)}(t)\right]^2}{M},$$
$$x_2^{(3)}(t) = -z_1x_2^{(2)}(t) - z_2x_3^{(2)}(t) + 2z_3x_1^{(1)}(t)\left[x_3^{(1)}(t) - x_1^{(1)}(t)\right] + z_3x_1(t)\left[x_3^{(2)}(t) - x_1^{(2)}(t)\right] + z_3x_1^{(2)}(t)\left[N - x_1(t) + x_3(t)\right],$$
$$x_3^{(3)}(t) = s_1s_2x_3(t)x_1^{(2)}(t) + 2s_1s_2x_1^{(1)}(t)x_3^{(1)}(t) + s_1x_3^{(2)}(t)\left[s_2x_1(t) - s_3\right],$$
$$x_4^{(3)}(t) = d_1x_1^{(2)}(t) - d_2x_4^{(2)}(t).$$

Expressions for the fourth-order derivatives of the solution functions:

$$x_1^{(4)}(t) = a_1x_1^{(3)}(t)\left[1 - \frac{2x_1(t)}{M}\right] - a_2\left[x_2^{(3)}(t) + x_3^{(3)}(t)\right] - d_3x_4^{(3)}(t) - \frac{6a_1x_1^{(1)}(t)x_1^{(2)}(t)}{M},$$
$$x_2^{(4)}(t) = -z_1x_2^{(3)}(t) - z_2x_3^{(3)}(t) + 3z_3x_1^{(2)}(t)\left[x_3^{(1)}(t) - x_1^{(1)}(t)\right] + 3z_3x_1^{(1)}(t)\left[x_3^{(2)}(t) - x_1^{(2)}(t)\right] + z_3x_1(t)\left[x_3^{(3)}(t) - x_1^{(3)}(t)\right] + z_3x_1^{(3)}(t)\left[N - x_1(t) + x_3(t)\right],$$
$$x_3^{(4)}(t) = s_1s_2x_3(t)x_1^{(3)}(t) + 3s_1s_2x_1^{(1)}(t)x_3^{(2)}(t) + 3s_1s_2x_1^{(2)}(t)x_3^{(1)}(t) + s_1x_3^{(3)}(t)\left[s_2x_1(t) - s_3\right],$$
$$x_4^{(4)}(t) = d_1x_1^{(3)}(t) - d_2x_4^{(3)}(t).$$

Expressions for the fifth-order derivatives of the solution functions:

$$x_1^{(5)}(t) = a_1\left(1 - \frac{2x_1(t)}{M}\right)x_1^{(4)}(t) - a_2\left(x_2^{(4)}(t) + x_3^{(4)}(t)\right) - d_3x_4^{(4)}(t) - \frac{8a_1x_1^{(1)}(t)x_1^{(3)}(t)}{M} - \frac{6a_1\left[x_1^{(2)}(t)\right]^2}{M},$$

$$x_2^{(5)}(t) = -z_1 x_2^{(4)}(t) - z_2 x_3^{(4)}(t) + 4z_3 \left[x_3^{(1)}(t) - x_1^{(1)}(t)\right] x_1^{(3)}(t)$$
$$+ 6z_3 \left[x_3^{(2)}(t) - x_1^{(2)}(t)\right] x_1^{(2)}(t) + 4z_3 \left[x_3^{(3)}(t) - x_1^{(3)}(t)\right] x_1^{(1)}(t)$$
$$+ z_3 \left[x_3^{(4)}(t) - x_1^{(4)}(t)\right] x_1(t) + z_3 \left[N - x_1(t) + x_3(t)\right] x_1^{(4)}(t),$$
$$x_3^{(5)}(t) = s_1 s_2 x_3(t) x_1^{(4)}(t) + 4s_1 s_2 x_1^{(1)}(t) x_3^{(3)}(t) + 6s_1 s_2 x_1^{(2)}(t) x_3^{(2)}(t)$$
$$+ 4\mathrm{s}_1 s_2 x_1^{(3)}(t) x_3^{(1)}(t) + s_1 \left[s_2 x_1 - s_3\right] x_3^{(4)}(t),$$
$$x_4^{(5)}(t) = d_1 x_1^{(4)}(t) - d_2 x_4^{(4)}(t).$$

Expressions for the sixth-order derivatives of the solution functions:

$$x_1^{(6)}(t) = a_1 \left[1 - \frac{2x_1(t)}{M}\right] x_1^{(5)}(t) - a_2 \left[x_2^{(5)}(t) + x_3^{(5)}(t)\right] - d_3 x_4^{(5)}(t)$$
$$- 10a_1 \frac{x_1^{(1)}(t) x_1^{(4)}(t)}{M} - 20a_1 \frac{x_1^{(2)}(t) x_1^{(3)}(t)}{M},$$
$$x_2^{(6)}(t) = -z_1 x_2^{(5)}(t) - z_2 x_3^{(5)}(t) + 5z_3 \left[x_3^{(1)}(t) - x_1^{(1)}(t)\right] x_1^{(4)}(t)$$
$$+ 10z_3 \left[x_3^{(2)}(t) - x_1^{(2)}(t)\right] x_1^{(3)}(t) + 10\mathrm{z}_3 \left[x_3^{(3)}(t) - x_1^{(3)}(t)\right] x_1^{(2)}(t)$$
$$+ 5z_3 \left[x_3^{(4)}(t) - x_1^{(4)}(t)\right] x_1^{(1)}(t) + z_3 \left[x_3^{(5)}(t) - x_1^{(5)}(t)\right] x_1(t)$$
$$+ z_3 \left[N - x_1(t) + x_3(t)\right] x_1^{(5)}(t),$$
$$x_3^{(6)}(t) = s_1 s_2 x_3(t) x_1^{(5)}(t) + 5s_1 s_2 x_1^{(1)}(t) x_3^{(4)}(t) + 10s_1 s_2 x_1^{(2)}(t) x_3^{(3)}(t)$$
$$+ 10\mathrm{s}_1 s_2 x_1^{(3)}(t) x_3^{(2)}(t) + 5s_1 s_2 x_1^{(4)}(t) x_3^{(1)}(t) + s_1 \left[s_2 x_1(t) - s_3\right] x_3^{(5)}(t),$$
$$x_4^{(6)}(t) = d_1 x_1^{(5)}(t) - d_2 x_4^{(5)}(t).$$

The error of the Taylor series numerical method is determined based on the order of the derivative. Specifically, a fourth-order Taylor series requires the use of derivative values up to the fourth order. In the theory of numerical methods, this method has a global error of $O(h^4)$ and a local error of $O(h^5)$. Similarly, a sixth-order Taylor series method has a global error of $O(h^6)$ and a local error of $O(h^7)$ [6, 10, 13].

3.1.2. Runge-Kutta Fourth Order Method

Among single-step numerical methods, the Runge-Kutta method is one of the most widely used approaches [17], particularly for solving scientific and engineering problems, which are often represented by nonlinear differential equation systems. Unlike the Taylor method, which requires computing high-order derivatives to approximate function values at a given point, the Runge-Kutta method approximates the solution using K intermediate stages [19]. The Runge-Kutta method includes both explicit and implicit forms [13–14]. In this study, we employ two commonly used Runge-Kutta formulas to solve the energy supply-demand system (2.1): the fourth-order and fifth-order Runge-Kutta methods. The fourth-order formula is presented below [10]:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right). \tag{3.2}$$

Where:

$\mathbf{k}_1 = h\mathbf{f}(t_i, \mathbf{x}_i)$,
$\mathbf{k}_2 = h\mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{1}{2}\mathbf{k}_1\right)$,
$\mathbf{k}_3 = h\mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{1}{2}\mathbf{k}_2\right)$,
$\mathbf{k}_4 = h\mathbf{f}\left(t_i + h, \mathbf{x}_i + \mathbf{k}_3\right)$.

The fourth-order formula utilizes four intermediate stages, k1 to k4, as defined above. This method has a global error of $O(h^4)$ and a local error of $O(h^5)$ [6, 10, 13].

3.1.3. Runge-Kutta Fifth Order Method

In this study, we implement the explicit fifth-order Runge-Kutta numerical method with high accuracy. This formula, proposed by Butcher, employs six intermediate stages to approximate the solutions of system (2.1) and is described as follows [20]:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{1}{90}(7\mathbf{k}_1 + 32\mathbf{k}_3 + 12\mathbf{k}_4 + 32\mathbf{k}_5 + 7\mathbf{k}_6)h. \tag{3.3}$$

Where:

$\mathbf{k}_1 = \mathbf{f}\left(t_i, \mathbf{x}_i\right)$,
$\mathbf{k}_2 = \mathbf{f}\left(t_i + \frac{1}{4}h, \mathbf{x}_i + \frac{1}{4}\mathbf{k}_1 h\right)$,
$\mathbf{k}_3 = \mathbf{f}\left(t_i + \frac{1}{4}h, \mathbf{x}_i + \frac{1}{8}\mathbf{k}_1 h + \frac{1}{8}\mathbf{k}_2 h\right)$,
$\mathbf{k}_4 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{x}_i - \frac{1}{2}\mathbf{k}_2 h + \mathbf{k}_3 h\right)$,
$\mathbf{k}_5 = \mathbf{f}\left(t_i + \frac{3}{4}h, \mathbf{x}_i + \frac{3}{16}\mathbf{k}_1 h + \frac{9}{16}\mathbf{k}_4 h\right)$,
$\mathbf{k}_6 = \mathbf{f}\left(t_i + h, \mathbf{x}_i - \frac{3}{7}\mathbf{k}_1 h + \frac{2}{7}\mathbf{k}_2 h + \frac{12}{7}\mathbf{k}_3 h - \frac{12}{7}\mathbf{k}_4 h + \frac{8}{7}\mathbf{k}_5 h\right)$.

This method has a global error of $O(h^5)$ and a local error of $O(h^6)$ [6, 10, 13].

3.2. Multi step methods

Unlike single-step numerical methods, multi-step numerical methods are designed to determine approximate solutions of a differential equation system at time t_{i+1} not only based on the solution at t_i but also by utilizing information from the solutions of the system at multiple previous time steps [6]. These methods include explicit, implicit, and a combined form of both, known as the predictor-corrector method. In this study, we implement one of the representative formulas of this approach, the Adams family of formulas, to solve system (2.1) [10].

Similar to single-step numerical methods, the accuracy of a multi-step numerical method is evaluated based on its order. A method of order p has a local accuracy of $O(h^{p+1})$ and a global accuracy of $O(h^p)$ [6, 10, 19].

3.2.1. Adams-Bashforth Methods

The Adams-Bashforth method is a typical explicit multi-step method [6, 13] used for solving nonlinear differential equation systems. In this study, we implement the fourth-order Adams-Bashforth formula to solve the energy supply-demand system. In this approach, the value of the solution function at any given time is approximated based on the four most recently computed solution values. Given a dataset containing j values $(t_i, \mathbf{f}_i)\,(t_{i-1}, \mathbf{f}_{i-1}), ..., (t_{i-j+1}, \mathbf{f}_{i-j+1})$, the fourth-order Adams-Bashforth formula ($j = 4$) is presented below [10]:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{24}\left[55\mathbf{f}_i - 59\mathbf{f}_{i-1} + 37\mathbf{f}_{i-2} - 9\mathbf{f}_{i-3}\right]. \tag{3.4}$$

To implement this method, the first step requires knowing the four initial solution values. Therefore, to determine the solution values from $\mathbf{x}_1$ to $\mathbf{x}_4$, a single -step method must be used. In this study, these initial values are obtained using the RK4 method.

3.2.2. Adams-Moulton Methods

The Adams-Moulton methods are an implicit multistep approach [13, 14]. This method is similar to the Adams-Bashforth method; however, the solution values are approximated not only based on previous steps but also on the predicted step at t_{i+1}. Considering a dataset with $j+1$ points $(t_{i+1}, \mathbf{f}_{i+1}), (t_i, \mathbf{f}_i), (t_{i-1}, \mathbf{f}_{i-1}), ..., (t_{i-j+1}, \mathbf{f}_{i-j+1})$ with $j = 3$ the fourth-order Adams-Moulton formula is given as follows [10]:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{24}\left[9\mathbf{f}_{i+1} + 19\mathbf{f}_i - 5\mathbf{f}_{i-1} + \mathbf{f}_{i-2}\right]. \tag{3.5}$$

3.2.3. Predictor-Corrector Methods

The Predictor-Corrector numerical method is a multi-step technique that combines both explicit and implicit multi-step methods. This approach consists of two main steps [13, 14]:

Predictor: Use an explicit multi-step numerical method to approximate the solution at step t_{i+1}

Corrector: The approximated solution obtained from the predictor step at t_{i+1} is then substituted into an implicit method to refine the approximation.

In this study, we implement the fourth-order Adams-Bashforth-Moulton Predictor-Corrector numerical method to solve system (2.1), where the fourth-order Adams-Bashforth formula is used in the Predictor step, and the fourth-order Adams-Moulton formula is used in the Corrector step. Additionally, the four initial approximate solution values are determined using the RK4 method. The specific formulas are presented below [10]:

Predictor P: Fourth-order Adams-Bashforth method.

$$\mathbf{x}_{i+1}^{(p)} = \mathbf{x}_i + \frac{h}{24}\left[55\mathbf{f}_i - 59\mathbf{f}_{i-1} + 37\mathbf{f}_{i-2} - 9\mathbf{f}_{i-3}\right]. \tag{3.6}$$

Corrector C: Fourth-order Adams-Moulton method.

$$\mathbf{x}_{i+1}^{(c)} = \mathbf{x}_i + \frac{h}{24}\left[9\mathbf{f}\left(t_{i+1}, \mathbf{x}_{i+1}^{(p)}\right) + 19\mathbf{f}_i - 5\mathbf{f}_{i-1} + \mathbf{f}_{i-2}\right]. \tag{3.7}$$

3.3. Adaptive numerical methods

One of the key techniques in solving nonlinear differential equation systems is the use of numerical methods capable of automatically adapting the step size flexibly. These numerical methods typically combine pairs of higher-order and lower-order methods to estimate errors and adjust the step size accordingly for different solution regions [21]. This approach ensures the maintenance of accuracy while simultaneously enhancing computational efficiency.

In this study, we implement adaptive step-size numerical methods supported by the `solve_ivp` library [18], including the RK45 method [11, 12]; the DOP853 method [13]; the Radau method [14]; the BDF method [15]; and the LSODA method [16]. To adjust the

adaptive step size, these numerical methods utilize two critical error tolerance parameters: Relative tolerances (rtol) and Absolute tolerances (atol), ensuring that the estimated local error of the solution satisfies the following condition [18]:

$$Local\_error < atol + rtol \times |\mathbf{x}| .$$

where:

$Local\_error$: is the estimated value of the local error.

$rtol$: is the relative tolerance.

$atol$: is the absolute tolerance.

$\mathbf{x}$: represents the approximate solutions of the differential equation system at the considered time point.

4. Evaluation Method

Since the exact solution of the nonlinear energy resources supply-demand system is unknown, evaluating numerical methods in this case is a challenging task. In this study, we propose several approaches to assess the numerical methods implemented as follows:

a) Evaluating the accuracy of numerical methods by comparing the residual between the left-hand side and the right-hand side:

To implement this method, the first task is to approximate the left-hand side of system (2.1). In this study, based on the discrete solutions obtained from numerical methods, we construct a first-order derivative approximation of the solution functions at the solution points. Several approaches can be used for this purpose, such as finite difference methods or function approximation. In this study, we approximate the system's solution functions using high-order spline interpolation [21] based on the discrete solution points, then compute the approximate derivatives at these points. Simultaneously, the obtained solutions are substituted into the right-hand side of the system. The Mean Absolute Error (MAE) method [9] is used to measure the discrepancy between the left-hand side and the right-hand side. A smaller MAE value indicates that the obtained solutions better satisfy system (2.1). The formula below describes this approach:

$$\mathbf{e}_{residual} = \frac{1}{L} \sum_{t_i \in T} |\mathbf{x}'_i - \mathbf{f}\left(t_i, \mathbf{x}_i\right)|. \tag{4.1}$$

Where:

$\mathbf{e}_{residual}$: is the residual error to be computed for the system's solutions.

$\mathbf{x}'_i$: represents the derivative approximation of the solution functions of the system at t_i, obtained using numerical solutions.

L: is the total number of solution points considered.

$T = (t_0, t_1, ..., t_{L-1})$: is the set of selected time points.

b) Evaluating the accuracy of numerical methods using reference solutions:

One of the limitations of the residual-based evaluation method described above is that the approximation of derivatives on the left-hand side may still introduce errors, depending on the approximation method used. Consequently, this evaluation approach may not be entirely accurate. In this study, we incorporate an additional evaluation method by using reference solutions. According to numerical method error theory, a higher-order numerical method or one using a smaller step size theoretically yields lower errors [6, 10, 13]. Based on

this principle, we use the solutions obtained from higher-order methods as reference solutions to assess the accuracy of solutions derived from lower-order methods or methods with larger step sizes. Specifically, in this study, for fixed-step methods, the solutions obtained from fourth-order methods are compared with those obtained from higher-order methods, namely RK5 and sixth-order Taylor. A fourth-order method whose solutions closely match those of higher-order methods is theoretically considered to have better accuracy.

For numerical methods with adaptive step sizes, we select a method and set its absolute and relative error tolerances, as well as its maximum step size, to very low values. This ensures that the solutions obtained using this method are theoretically more accurate. These solutions are then used as reference solutions to evaluate the accuracy of numerical methods configured with higher absolute and relative error tolerances, as well as larger maximum step sizes. The Mean Absolute Error (MAE) method is then employed to evaluate the errors, and its mathematical formulation is as follows:

$$\mathbf{e}_{error} = \frac{1}{L} \sum_{t_i \in T} \left| \mathbf{x}_{i\_lower} - \mathbf{x}_{i\_higher} \right|. \tag{4.2}$$

Where:

$\mathbf{e}_{error}$: The error between the lower-accuracy method and the higher-accuracy method.
$\mathbf{x}_{i\_lower}$: The solution of the method with theoretically lower accuracy at t_i
$\mathbf{x}_{i\_higher}$: The solution of the method with theoretically higher accuracy at t_i
L: The total number of solution points considered.
$T = (t_0, t_1, ..., t_{L-1})$: is the set of selected time points.

c) Convergence Rate Evaluation:

In this study, we employ numerical methods of the same fourth order to compare their convergence rates. This is achieved by solving system (2.1) using these methods with progressively smaller step sizes. Theoretically, as the step size decreases, the obtained solution becomes more accurate. Therefore, if a method's solution at a larger step size rapidly approaches its solution at smaller step sizes, that method exhibits a faster empirical convergence rate. We apply the following formula to evaluate the convergence rate of these methods:

$$C_{speed} = \frac{\frac{1}{L_1} \sum\limits_{t_i \in T_1} \left| \mathbf{x}_{i\_h_1} - \mathbf{x}_{i\_h_2} \right|}{\frac{1}{L_2} \sum\limits_{t_i \in T_2} \left| \mathbf{x}_{i\_h_2} - \mathbf{x}_{i\_h_3} \right|}. \tag{4.3}$$

Where:

C_{speed}: The index for evaluating the convergence rate of the method
$\mathbf{x}_{i\_h_1}$: Solutions obtained by the numerical method with step size h_1 at time t_i
$\mathbf{x}_{i\_h_2}$: Solutions obtained by the numerical method with step size h_2 at time t_i
$\mathbf{x}_{i\_h_3}$: Solutions obtained by the numerical method with step size h_3 at time t_i
L_1: Total number of solution points compared between the numerical method using step sizes h_1 and h_2
$T_1 = (t_0, t_1, ..., t_{L_1-1})$: the set of time points used to compare the solutions obtained by the numerical method using two different step sizes, h_1 and h_2.
L_2: Total number of solution points compared between the numerical method using step sizes h_2 and h_3

$T_2 = (t_0, t_1, ..., t_{L_2-1})$: the set of time points used to compare the solutions obtained by the numerical method using two different step sizes, h_2 and h_3.

For this evaluation method, the numerical methods being compared must have the same order and be solved using the same step sizes h_1, h_2, and h_3, satisfying the condition $h_1 > h_2 > h_3$. In this study, we use step sizes that decrease uniformly, specifically $h_2 = h_1/2$; $h_3 = h_2/2$. A method with a higher C_{speed} value indicates a faster convergence rate within the considered step size range.

5. Results

In this study, we conduct experiments to solve the differential equation system (2.1), using the parameter values and initial conditions provided in Section 2., over the interval $t = [0, 5000]$. Table 5.1 presents the accuracy of fixed-step numerical methods using the residual comparison approach based on formula 4.1, with the step size set to $h = 0.01$, thereby obtaining $L = 500,000$ discrete solution points. For this evaluation method, the step size must be sufficiently small to ensure a more accurate approximation of the derivatives on the left-hand side. In this study, from the discrete solution set obtained using numerical methods, we apply a fifth-degree spline function to approximate the unknown functions [21], then compute the approximate derivatives of these functions at the solution points. This allows us to determine the approximated values on the left-hand side of the system.

The results indicate that single-step numerical methods, such as Runge-Kutta and Taylor methods, achieve higher accuracy compared to multi-step methods, including Adams-Bashforth and Adams Predictor-Corrector. Among them, although both exhibit similar accuracy, the Taylor method requires less computation time than the Runge-Kutta method. Additionally, although the RK5 method has a lower order than the sixth-order Taylor method, it has a longer computation time. However, RK5 can achieve comparable accuracy to the sixth-order Taylor method, suggesting that despite its higher computational cost, it exhibits stability and effective error control. The Adams-Bashforth method demonstrates the lowest accuracy but is also the most computationally efficient.

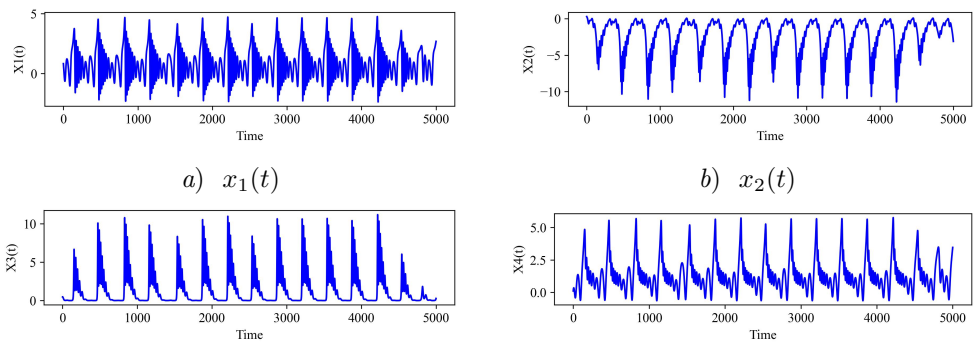
Таблица 5.1. Оценка погрешности численных методов с фиксированным шагом при h = 0.01 на основе метода сравнения остатка

Table 5.1. Error Evaluation of Fixed-Step Numerical Methods with h = 0.01 based on the Residual Comparison Approach.

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK4	2.32198×10^{-12}	1.95121×10^{-12}	2.19175×10^{-12}	1.08941×10^{-12}	12.508
RK5	2.14742×10^{-12}	1.87919×10^{-12}	2.15478×10^{-12}	1.08296×10^{-12}	25.666
Taylor Order 4	2.22146×10^{-12}	2.44939×10^{-12}	2.80666×10^{-12}	1.09848×10^{-12}	9.255
Taylor Order 6	2.14760×10^{-12}	1.87941×10^{-12}	2.15499×10^{-12}	1.08320×10^{-12}	14.861
Adams-BF 4	6.41036×10^{-10}	3.42839×10^{-10}	6.61115×10^{-10}	2.00785×10^{-09}	5.575
Adams-PC 4	2.18936×10^{-10}	1.10598×10^{-10}	2.18661×10^{-10}	6.93564×10^{-10}	10.037

Figures 5.1, 5.2, 5.3, and 5.4 respectively illustrate the numerical solutions of system (2.1) obtained using fourth-order methods. The numerical solutions exhibit strong oscillations

with the presence of high-frequency regions, indicating the system's strong nonlinear characteristics. If the step size h is not sufficiently small, the method may fail to capture the rapid changes in the solution, leading to significant error accumulation as the solution domain t increases. A distinction can be observed between the solutions obtained from single-step methods, namely RK4 and the fourth-order Taylor method, and those obtained from multi-step methods, such as Adams-Bashforth and Adams Predictor-Corrector.

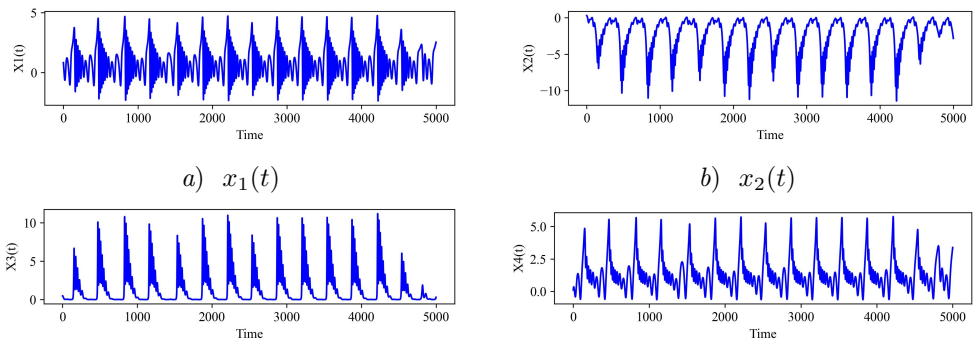


a) $x_1(t)$ *b)* $x_2(t)$

c) $x_3(t)$ *d)* $x_4(t)$

Рис. 5.1. Численные решения системы (2.1), полученные с использованием метода РК4 при шаге $h = 0.01$

Fig. 5.1. Numerical solutions of the system (2.1), obtained using the RK4 method with $h = 0.01$



a) $x_1(t)$ *b)* $x_2(t)$

c) $x_3(t)$ *d)* $x_4(t)$

Рис. 5.2. Численные решения, полученные с использованием метода Тейлора четвёртого порядка при шаге $h = 0.01$

Fig. 5.2. Numerical solutions obtained using the fourth-order Taylor method with $h = 0.01$

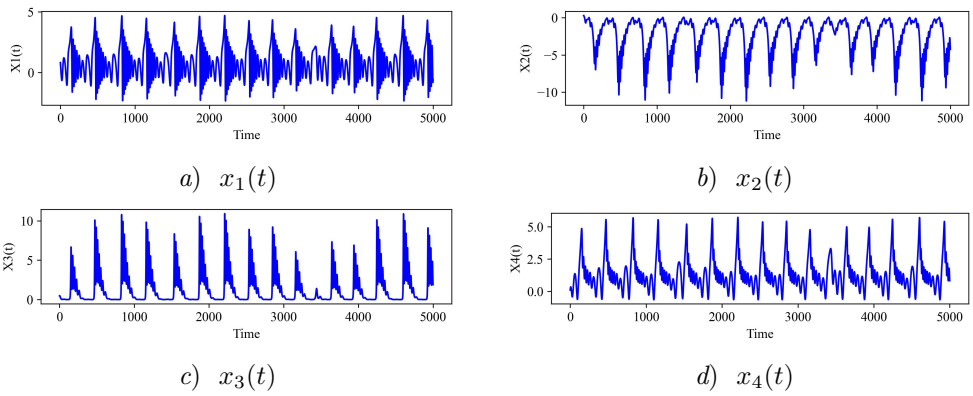


Рис. 5.3. Численные решения, полученные с использованием метода Адамса — Башфорта четвёртого порядка при шаге $h = 0.01$

Fig. 5.3. Numerical solutions obtained using the fourth-order Adams-Bashforth method with $h = 0.01$

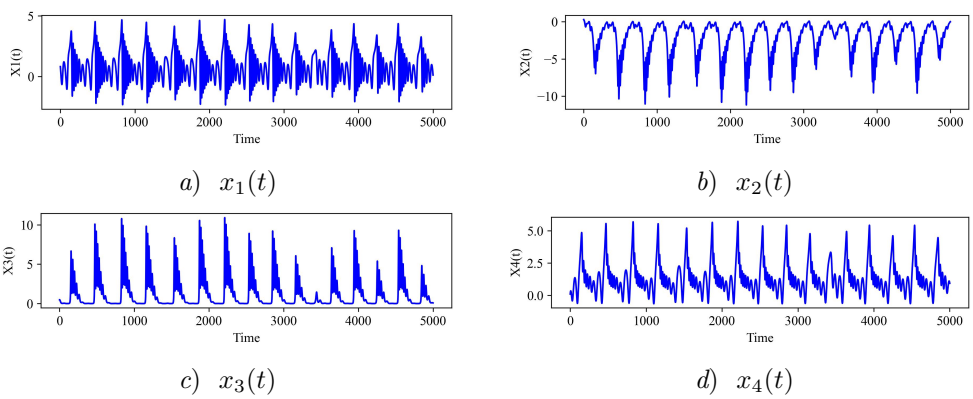


Рис. 5.4. Численные решения, полученные с использованием метода Адамса прогноза–коррекции четвёртого порядка при шаге $h = 0.01$

Fig. 5.4. Numerical solutions obtained using the fourth-order Adams Predictor-Corrector method with $h = 0.01$

Figures 5.5 and 5.6 illustrate the numerical solutions of system (2.1) obtained using the higher-order methods RK5 and sixth-order Taylor. It is observed that the numerical solutions obtained from both methods exhibit minimal differences, indicating that with a step size of $h = 0.01$, the solutions are relatively convergent over the interval $t = [0, 5000]$.

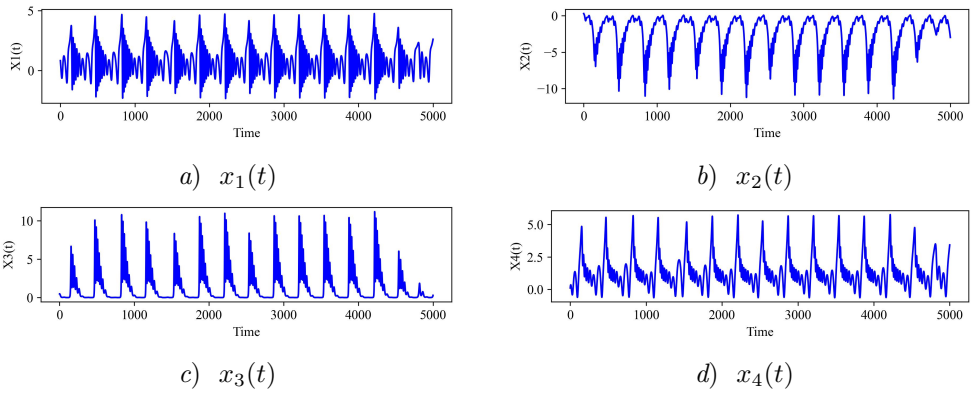


Рис. 5.5. Численные решения, полученные с использованием метода РК5 при шаге $h = 0.01$

Fig. 5.5. Numerical solutions obtained using the RK5 method with $h = 0.01$

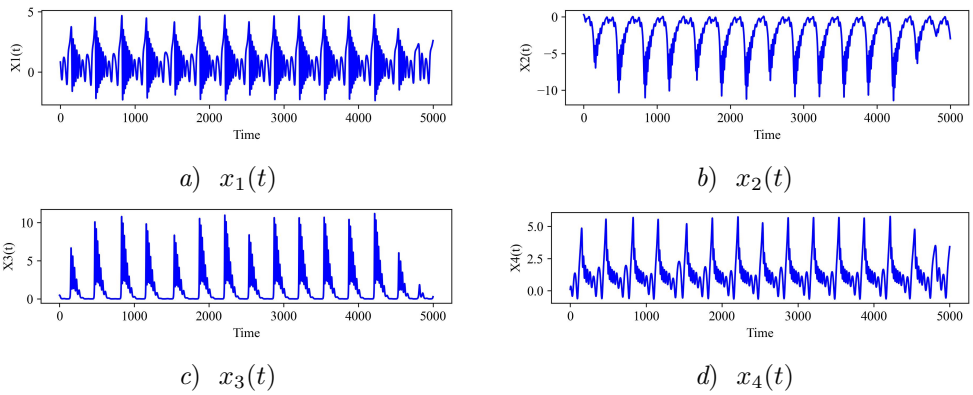


Рис. 5.6. Численные решения, полученные с использованием метода Тейлора шестого порядка при шаге $h = 0.01$

Fig. 5.6. Numerical solutions obtained using the sixth-order Taylor method with $h = 0.01$

Figures 5.7 and 5.8 illustrate the numerical solutions of system (2.1) using the high-order methods RK5 and sixth-order Taylor with a step size of $h = 0.01$. It is observed that as the time domain extends to $t = [0, 10000]$, the accumulated error increases significantly, leading to considerable discrepancies between the numerical solutions obtained by the two methods. Therefore, for the examined time domain, reducing the step size may be necessary to ensure greater accuracy and convergence of the solution.

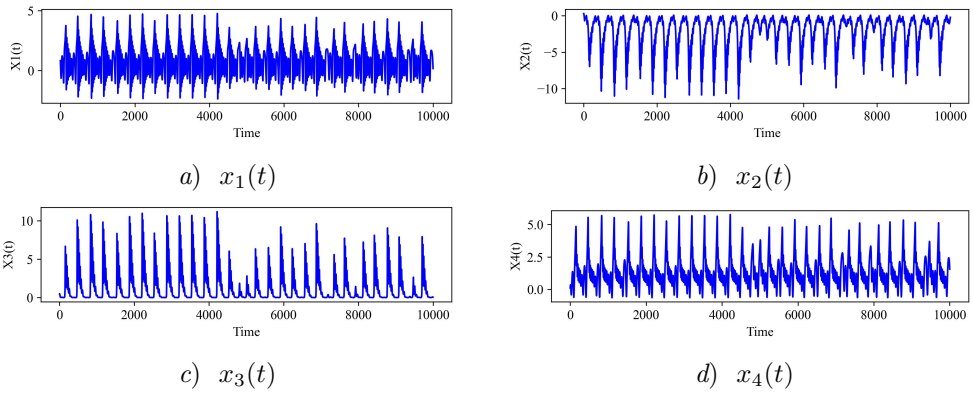


Рис. 5.7. Численные решения, полученные с использованием метода РК5 при шаге $h = 0.01$ и $t = [0, 10000]$

Fig. 5.7. Numerical solutions obtained using the RK5 method with $h = 0.01$ and $t = [0, 10000]$

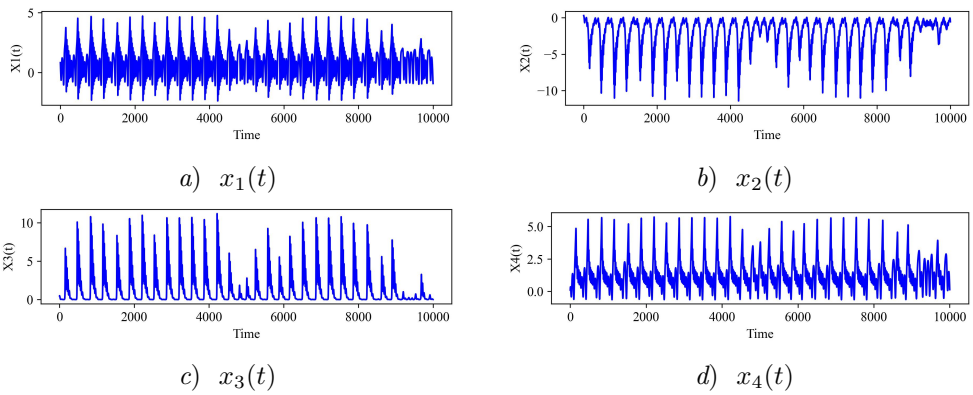
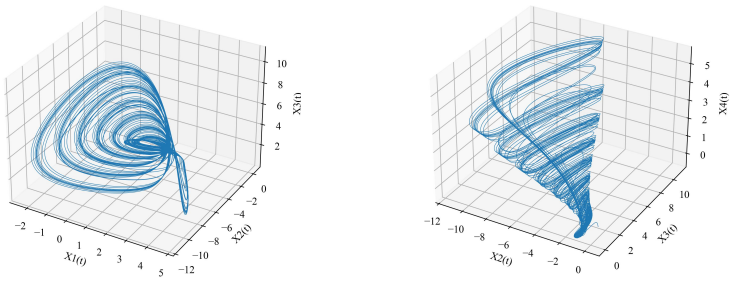


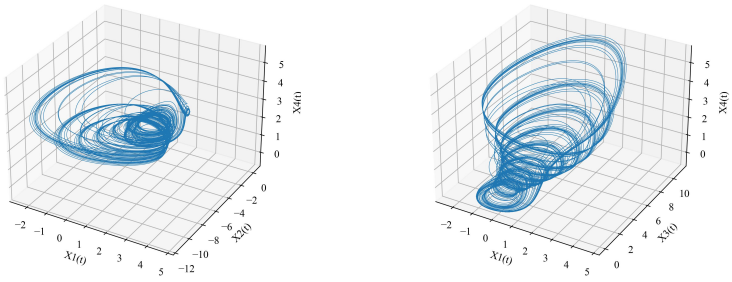
Рис. 5.8. Численные решения, полученные с использованием метода Тейлора шестого порядка при шаге $h = 0.01$ и $t = [0, 10000]$

Fig. 5.8. Numerical solutions obtained using the sixth-order Taylor method with $h = 0.01$ and $t = [0, 10000]$

Figures 5.9 illustrate the numerical solutions of system (2.1) in a three-dimensional space, obtained using the sixth-order Taylor method, with a step size of $h = 0.01$ and $t = [0, 5000]$. It is observed that the system's solutions in the chaotic state exhibit a complex dynamical structure.



a) $Ox_1x_2x_3$ *b)* $Ox_2x_3x_4$



c) $Ox_1x_2x_4$ *d)* $Ox_1x_3x_4$

Рис. 5.9. Графики проекций фазовых траекторий системы (2.1), полученные численным методом Тейлора шестого порядка при $h = 0.01$ и $t = [0, 5000]$, на соответствующие трехмерные подпространства

Fig. 5.9. Graphs of projections of phase trajectories of the system (2.1), obtained by the sixth-order Taylor numerical method for $h = 0.01$ and $t = [0, 5000]$, onto the corresponding three-dimensional subspaces

Tables 5.2 and 5.3 present the results of comparing fourth and fifth-order fixed-step numerical methods with the higher-order sixth-order Taylor method, using formula 4.2 over the solution domain $t = [0, 5000]$ at different step sizes of 0.1 and 0.01. The results indicate that, compared to fourth-order methods, the solutions obtained from the fifth-order RK5 method exhibit significantly lower error when compared to the sixth-order Taylor method.

When comparing the numerical solutions obtained from different fourth-order methods with those from the sixth-order method, the results indicate that single-step fourth-order methods yield lower errors than multi-step methods. Among them, the fourth-order Taylor and fourth-order Runge-Kutta methods demonstrate comparable error levels, but the fourth-order Taylor method requires less computation time than the fourth-order Runge-Kutta method. However, at a larger step size ($h = 0.1$), the RK4 method produces solutions with lower error than the fourth-order Taylor method. As the step size decreases, the results from Tables 5.2 and 5.3 further show that the single-step methods RK4 and fourth-order Taylor exhibit a faster error reduction rate compared to multi-step methods based on the Adams

formula. Consequently, their solutions converge more rapidly to the sixth-order method's solutions.

Таблица 5.2. Сравнение погрешностей численных методов четвёртого и пятого порядка с методом Тейлора шестого порядка при шаге $h = 0.1$
Table 5.2. Error comparison of fourth-order and fifth-order numerical methods with the sixth-order Taylor method at $h = 0.1$

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK4	0.3201646816	0.3074583273	0.2545435753	0.2405862251	1.301
RK5	0.0106139637	0.0066821595	0.0025826036	0.0091452239	2.548
Taylor order 4	0.3677339589	0.5283631888	0.3963504489	0.3275716874	0.933
Adams-BF 4	0.9803687400	1.0451732467	0.8478472690	0.7701285329	0.584
Adams-PC 4	0.9257142090	1.0183955740	0.7794342566	0.7509303939	1.016

Таблица 5.3. Сравнение погрешностей численных методов четвёртого и пятого порядка с методом Тейлора шестого порядка при шаге $h = 0.01$
Table 5.3. Error comparison of fourth-order and fifth-order numerical methods with the sixth-order Taylor method at $h = 0.01$

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK4	0.0016633238	0.0011119776	0.0004133325	0.0013543109	12.508
RK5	8.11004×10^{-05}	5.47477×10^{-05}	2.01610×10^{-05}	6.55285×10^{-05}	25.666
Taylor order 4	0.0013549301	0.0009225286	0.0003366501	0.0010875024	9.255
Adams-BF 4	0.6969048783	0.9875354755	0.7338413482	0.6110239391	5.575
Adams-PC 4	0.6352375876	0.8482451456	0.6250611408	0.5381780426	10.037

Tables 5.4 and 5.5 present the errors between the numerical solutions obtained using fourth-order methods and the fifth-order Runge-Kutta method over the solution domain $t = [0, 5000]$, at step sizes of 0.1 and 0.01. The results exhibit similar trends to those observed when comparing these methods with the sixth-order Taylor method.

Таблица 5.4. Сравнение погрешностей между численными методами четвёртого порядка и методом Рунге-Кутты пятого порядка (RK5) при шаге $h = 0.1$
Table 5.4. Comparison of errors between fourth-order numerical methods and RK5 at $h = 0.1$

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK4	0.3222448445	0.3093120751	0.2551649732	0.2409296628	1.301
Taylor order 4	0.3686927666	0.5287241577	0.3957364282	0.3253967559	0.933
Adams-BF 4	0.9800879315	1.0470551703	0.8487057242	0.7690135558	0.584
Adams-PC 4	0.9219998790	1.0193316583	0.7804187325	0.7464283058	1.016

Таблица 5.5. Сравнение погрешностей между численными методами четвёртого порядка и методом Рунге-Кутты пятого порядка (RK5) при шаге $h = 0.01$
Table 5.5. Comparison of errors between fourth-order numerical methods and RK5 at $h = 0.01$

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK4	0.0015822653	0.0010573021	0.0003931734	0.0012888379	12.508
Taylor order 4	0.0014360006	0.0009772263	0.0003568097	0.0011529920	9.255
Adams-BF 4	0.6969151280	0.9875166627	0.7338327435	0.6110366128	5.575
Adams-PC 4	0.6352336703	0.8482619023	0.6250628251	0.5381703950	10.037

To evaluate the convergence rate of numerical methods, this study applies various fourth-order numerical methods to solve system (2.1) over the solution domain $t = [0, 5000]$, with progressively decreasing step sizes, starting from $h = 0.5$. The error between numerical solutions at successive step size reductions (each step size being half of the previous one) is measured, and the convergence rate is computed using formula 4.3 proposed in this study. The results, presented in Table 5.6, show that for larger initial step sizes, all numerical methods exhibit similar convergence rates, mostly around 1. This indicates that all the methods converge slowly or have not yet stabilized within the considered step size range.

Tables 5.7 and 5.8 present results for further reductions in step size, showing that single-step methods demonstrate an increasing convergence rate, which becomes significantly higher as the step size decreases. In contrast, multi-step methods exhibit a slower and inconsistent convergence rate. These findings indicate that single-step methods tend to converge more effectively than multi-step methods, while multi-step methods may require smaller step sizes to achieve convergence. This result is also consistent with previous evaluations based on the residual method and reference solutions obtained from higher-order numerical methods.

Таблица 5.6. Сравнение скоростей сходимости численных методов четвёртого порядка с фиксированным шагом при $h_1 = 0.5$, $h_2 = 0.25$, $h_3 = 0.125$
Table 5.6. Comparison of the convergence rates of fourth-order fixed-step methods at $h_1 = 0.5$, $h_2 = 0.25$, $h_3 = 0.125$

Method	X1(t) speed	X2(t) speed	X3(t) speed	X4(t) speed
RK4	1.485	1.132	1.209	1.445
Taylor order 4	1.235	1.042	1.174	1.124
Adams-BF 4	1.285	1.369	1.383	1.269
Adams-PC 4	1.047	0.769	0.816	0.935

Таблица 5.7. Сравнение скоростей сходимости численных методов четвёртого порядка с фиксированным шагом при $h_1 = 0.125$, $h_2 = 0.0625$, $h_3 = 0.03125$
Table 5.7. Comparison of the convergence rates of fourth-order fixed-step methods at $h_1 = 0.125$, $h_2 = 0.0625$, $h_3 = 0.03125$

Method	X1(t) speed	X2(t) speed	X3(t) speed	X4(t) speed
RK4	1.942	2.531	2.392	2.119
Taylor order 4	1.665	2.916	2.637	1.888
Adams-BF 4	1.166	1.427	1.363	1.268
Adams-PC 4	1.176	1.945	1.776	1.321

Таблица 5.8. Сравнение скоростей сходимости численных методов четвёртого порядка с фиксированным шагом при $h_1 = 0.03125$, $h_2 = 0.015625$, $h_3 = 0.0078125$

Table 5.8. Comparison of the convergence rates of fourth-order fixed-step methods at $h_1 = 0.03125$, $h_2 = 0.015625$, $h_3 = 0.0078125$

Method	X1(t) speed	X2(t) speed	X3(t) speed	X4(t) speed
RK4	8.040	8.849	11.401	8.316
Taylor order 4	7.196	8.843	16.467	6.634
Adams-BF 4	1.140	1.082	1.150	1.060
Adams-PC 4	1.162	1.169	1.175	1.177

In this study, adaptive step-size numerical methods supported by the SciPy library [17–18] are implemented to solve the differential equation system (2.1). These methods include RK45, DOP853, Radau, BDF, and LSODA. The system (2.1) is solved over the domain $t = [0, 5000]$, discretized into $L = 500,000$ equally spaced solution points.

Tables 5.9 present the accuracy evaluation of these methods using the residual-based measurement defined in equation 4.1, with the parameter max step = 0.01, $atol = 10^{-12}$ and $rtol = 10^{-09}$. The results indicate that adaptive single-step methods, such as RK45, DOP853, and Radau, achieve higher accuracy than adaptive multi-step methods, namely BDF and LSODA, with DOP853 exhibiting the best accuracy.

Таблица 5.9. Сравнение погрешностей адаптивных численных методов с использованием метода измерения остатка

Table 5.9. Comparison of the errors of adaptive numerical methods using the residual measurement method

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK45	1.62218×10^{-14}	3.18855×10^{-14}	1.81544×10^{-14}	1.93451×10^{-14}	54.624
DOP853	1.60911×10^{-14}	3.14930×10^{-14}	1.79521×10^{-14}	1.91034×10^{-14}	104.688
Radau	3.13656×10^{-14}	6.87510×10^{-14}	6.66809×10^{-14}	1.93717×10^{-14}	140.390
BDF	2.23207×10^{-12}	7.22017×10^{-12}	9.07787×10^{-12}	5.86248×10^{-13}	87.648
LSODA	2.33088×10^{-12}	2.10136×10^{-12}	2.35122×10^{-12}	1.14341×10^{-12}	18.278

In addition to evaluating the accuracy of adaptive numerical methods through residual error measurement between the left-hand and right-hand sides, this study also assesses these methods by comparing them with reference solutions, as presented in Section 4. Specifically, two methods were selected to generate reference solutions: DOP853, a high-order method representing single-step numerical methods, and BDF, representing multi-step numerical methods. These methods were assigned stringent accuracy parameters (i.e., small values for *atol*, *rtol* and max step). The system (2.1) is then solved using the two selected reference methods to obtain highly accurate solutions, which serve as reference solutions for comparison and evaluation. The adaptive numerical methods under assessment were assigned lower accuracy requirements.

Tables 5.10 and 5.11 present the errors when comparing the solutions of adaptive

numerical methods configured with accuracy settings of $atol = 10^{-09}$, $rtol = 10^{-06}$, and max step = 0.1 to the highly accurate reference solutions obtained using the selected reference methods, DOP853 and BDF ($atol = 10^{-12}$, $rtol = 10^{-09}$ and max step = 0.01), over the interval $t = [0, 5000]$. The results show that adaptive single-step numerical methods continue to outperform adaptive multi-step numerical methods. Among them, DOP853 achieves the highest accuracy, followed by the single-step methods Radau and RK45, which have comparable accuracy. However, Radau requires a longer computation time, which is entirely expected due to its implicit nature.

Таблица 5.10. Сравнение погрешностей между адаптивными численными методами с низкими настройками точности ($atol = 10^{-09}$, $rtol = 10^{-06}$ и максимальный шаг = 0.1) и эталонным методом DOP853, настроенным на высокую точность ($atol = 10^{-12}$, $rtol = 10^{-09}$, max step = 0.01)

Table 5.10. Error comparison between adaptive numerical methods with lower accuracy settings ($atol = 10^{-09}$, $rtol = 10^{-06}$ and max step=0.1) and the reference method DOP853, set with parameters for high accuracy ($atol = 10^{-12}$, $rtol = 10^{-09}$, max step = 0.01)

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK45	0.0847663554	0.0813651643	0.0571416993	0.0634416895	5.662
DOP853	1.23895×10^{-05}	8.36145×10^{-06}	3.07995×10^{-06}	1.00127×10^{-05}	10.392
Radau	0.0759208190	0.0708642717	0.0474347373	0.0559951801	13.920
BDF	0.6086937873	0.5833698577	0.5198981051	0.4354076134	8.677
LSODA	0.5265124258	0.4244004704	0.3941582924	0.3654465934	2.146

Таблица 5.11. Сравнение погрешностей между адаптивными численными методами с низкими настройками точности ($atol = 10^{-09}$, $rtol = 10^{-06}$ и максимальный шаг = 0.1) и эталонным методом BDF, настроенным на высокую точность ($atol = 10^{-12}$, $rtol = 10^{-09}$, max step = 0.01)

Table 5.11. Error comparison between adaptive numerical methods with lower accuracy settings ($atol = 10^{-09}$, $rtol = 10^{-06}$ and max step=0.1) and the reference method BDF, set with parameters for high accuracy ($atol = 10^{-12}$, $rtol = 10^{-09}$, max step = 0.01)

Method	X1(t) error	X2(t) error	X3(t) error	X4(t) error	Time (s)
RK45	0.1099220031	0.1186788941	0.0876441707	0.0886665677	5.662
DOP853	0.0668795589	0.0457344731	0.0317789621	0.0591959604	10.392
Radau	0.1023159919	0.1080904800	0.0780278565	0.0848352444	13.920
BDF	0.6065371580	0.6115335048	0.5396587152	0.4378548469	8.677
LSODA	0.5309210513	0.4370703576	0.4003046808	0.3784663871	2.146

Figures 5.10, 5.11, 5.12, 5.13, and 5.14 respectively depict the numerical solutions of system (2.1) using the adaptive numerical methods RK45, DOP853, Radau, BDF, and LSODA over the domain $t = [0, 5000]$, with max step = 0.01, $atol = 10^{-12}$, $rtol = 10^{-09}$.

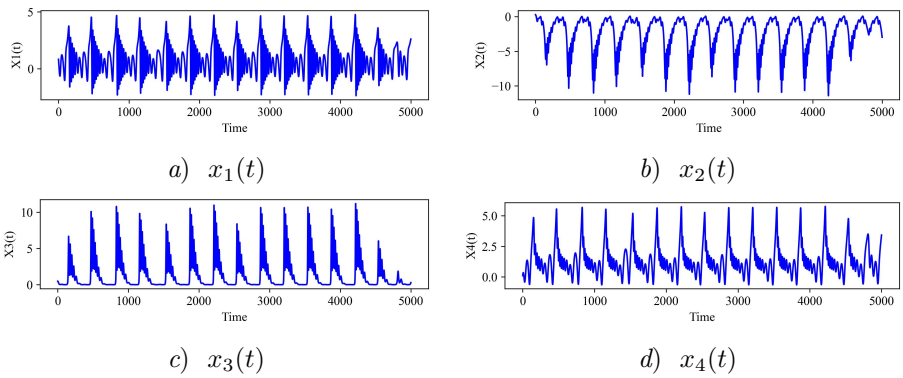


Рис. 5.10. Численные решения, полученные с использованием метода RK45

Fig. 5.10. Numerical solutions obtained using the RK45 method

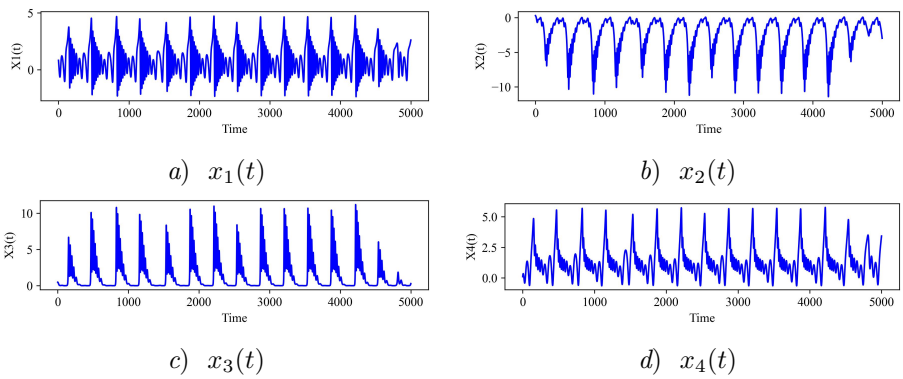


Рис. 5.11. Численные решения, полученные с использованием метода DOP853

Fig. 5.11. Numerical solutions obtained using the DOP853 method

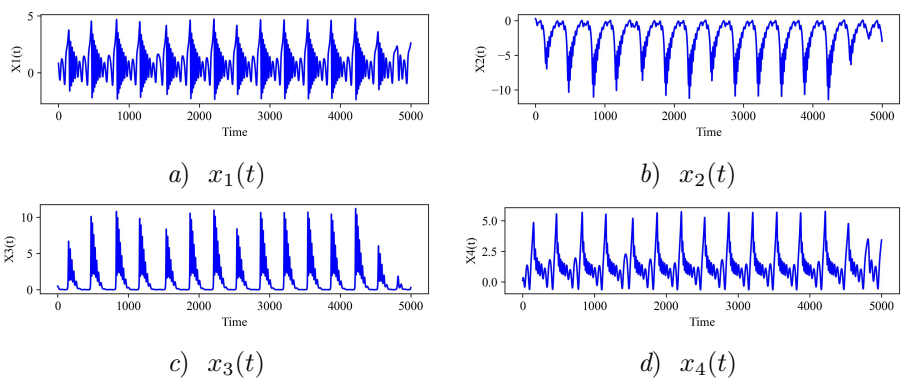


Рис. 5.12. Численные решения, полученные с использованием метода Radau

Fig. 5.12. Numerical solutions obtained using the Radau method

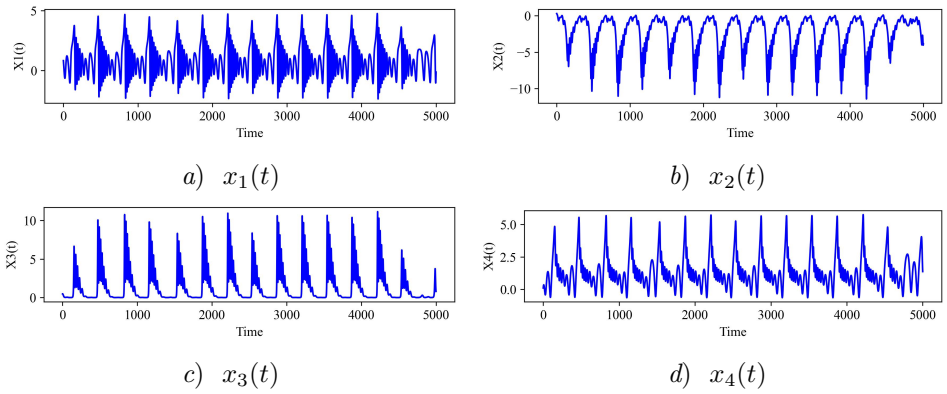


Рис. 5.13. Численные решения, полученные с использованием метода BDF

Fig. 5.13. Numerical solutions obtained using the BDF method

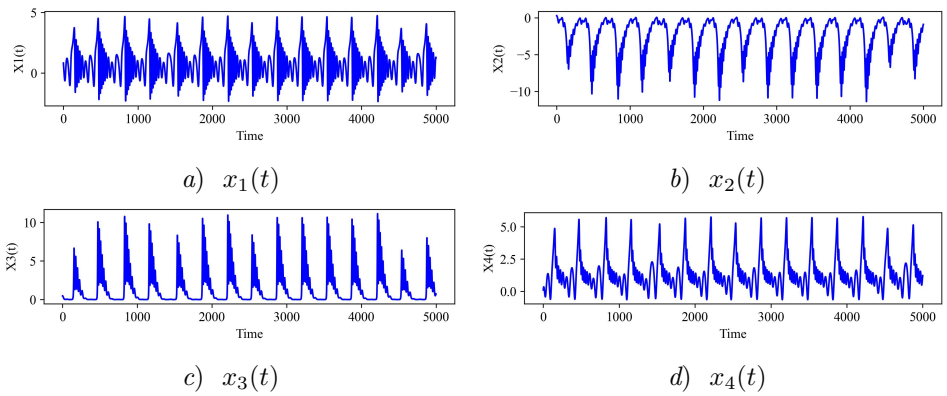


Рис. 5.14. Численные решения, полученные с использованием метода LSODA

Fig. 5.14. Numerical solutions obtained using the LSODA method

Figures 5.15 and 5.16 illustrate the numerical solutions of the system over the extended time domain $t = [0, 10000]$, obtained using two highly regarded numerical methods: DOP853 and Radau. It can be observed that as the time domain increases, the solutions no longer maintain the similarity seen in figures 5.11 and 5.12 (where $t \leq 5000$). Furthermore, a comparison of the numerical results in the domain $t > 5000$ obtained using the RK5 and sixth-order Taylor methods as shown in figures 5.7 and 5.8, reveals that when the system is in a chaotic state, the accumulated error increases rapidly, leading to significant discrepancies in the numerical solutions and making convergence difficult within this domain.

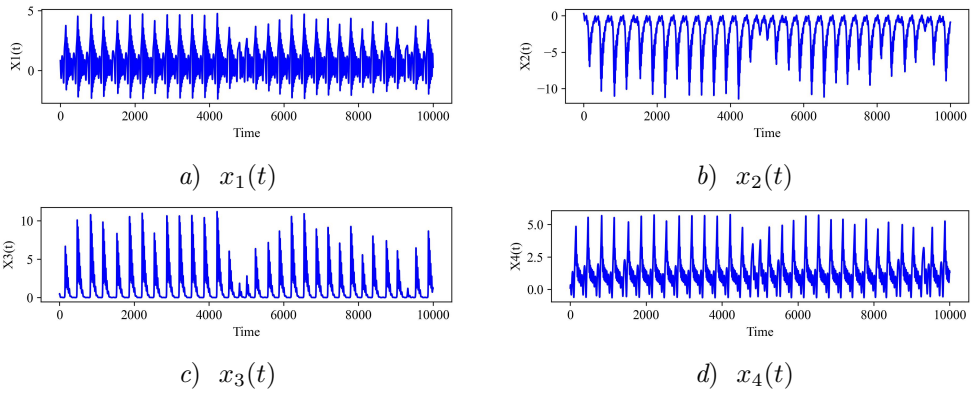


Рис. 5.15. Численные решения, полученные с использованием метода DOP853 при $t = [0, 10000]$

Fig. 5.15. Numerical solutions obtained using the DOP853 method with $t = [0, 10000]$

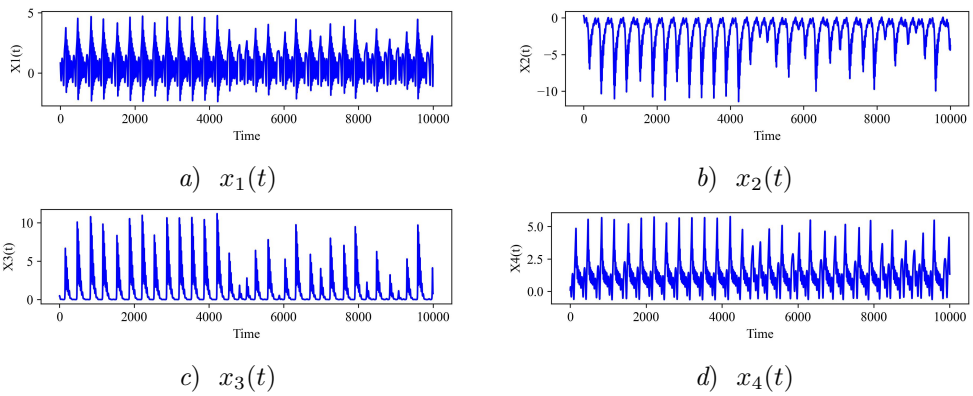


Рис. 5.16. Численные решения, полученные с использованием метода Radau при $t = [0, 10000]$

Fig. 5.16. Numerical solutions obtained using the Radau method with $t = [0, 10000]$

6. Conclusion

In this study, we implemented numerical methods to solve the nonlinear energy resources supply-demand system, including single-step methods, multi-step methods, and adaptive step-size methods. The effectiveness of these methods was analyzed and evaluated for the given problem. Experimental results indicate that, in the considered cases, single-step methods were more effective than multi-step methods in terms of accuracy and convergence speed, while multi-step methods demonstrated higher computational efficiency, as they required less computation time. Adaptive step-size numerical methods demonstrated both flexibility and stability. Based on the evaluation and analysis of the system's numerical solutions, the system exhibits nonlinearity and chaotic behavior. In addition, the system's

components change rapidly and influence each other significantly, potentially leading to irregular oscillations and the emergence of complex dynamics. Therefore, to ensure numerical stability and accurately simulate the long-term dynamic behavior of the system, it is essential to employ high-order numerical methods combined with an appropriate step size. In this study, we focused on implementing representative numerical methods to solve the nonlinear energy resources supply-demand system, in which certain aspects of the system's behavior were observed through numerical solutions. To gain a more comprehensive and in-depth understanding, future research should extend the experimental scope, investigate a wider range of solution domains and parameters, and analyze the effectiveness of numerical methods when applied to more complex cases of the system.

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