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On some universal criterion for a fixed point

A. V. Litavrin

Siberian Federal University (Krasnoyarsk, Russian Federation)

Abstract. Fixed point criteria may be applied in various fields of mathematics. The problem of finding sufficient conditions for transformations of a certain class to have a fixed point is well known. In the context of element-wise description for the monoid of all endomorphisms of a groupoid, the following were formulated: a bipolar classification of endomorphisms and related mathematical objects. In particular, the concept of a bipolar type of endomorphism of a groupoid (or simply a bipolar type) was stated. Every endomorphism of an arbitrary groupoid has exactly one bipolar type. In this paper, using bipolar types, a fixed point criterion for an arbitrary transformation of a nonempty set (hereinafter, the universal fixed point criterion) is formulated and proved. This criterion is not easy to apply. Further expansion of the range of problems to which this criterion can be applied depends directly on the success in studying the properties of groupoids' endomorphisms. The paper formulates such general problems (unsolved for today), that the success in their study will expand the possibilities of using the universal fixed point criterion. The connection between the formulated problems and the obtained criterion is discussed. In particular, necessary and sufficient conditions for the Riemann hypothesis on the zeros of the Riemann zeta function to be satisfied are obtained using the universal fixed point criterion.

Keywords: groupoid, endomorphism, bipolar type, bipolar classification of endomorphisms of a groupoid, fixed point, universal fixed point criterion, Riemann hypothesis on the zeros of the zeta function, Riemann zeta function

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About the authors:
Andrey V. Litavrin, Ph. D. in Physics and Mathematics, Associate Professor of the Department of Higher Mathematics No. 2, Siberian Federal University (79 Svobodny Av., Krasnoyarsk 660041, Russia), ORCID: https://orcid.org/0000-0001-6285-0201, anm11@rambler.ru

Оригинальная статья

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Об одном универсальном критерии неподвижной точки

А. В. Литаврин

Сибирский Федеральный Университет (г. Красноярск, Российская Федерация)

Аннотация. Критерии неподвижной точки находят применение в различных областях математики. Хорошо известен интерес к проблеме нахождения достаточных условий того, что преобразование из некоторого класса имеет неподвижную точку. В контексте изучения проблемы поэлементного описания моноида всех эндоморфизмов группоида были сформулированы: биполярная классификация эндоморфизмов и сопутствующие математические объекты. В частности, было сформулировано понятие «биполярный тип эндоморфизма» группоида (или просто «биполярный тип»). Всякий эндоморфизм произвольного группоида имеет ровно один биполярный тип. В данной работе с помощью биполярных типов формулируется и доказывается критерий неподвижной точки произвольного преобразования некоторого непустого множества (далее универсальный критерий неподвижной точки). Данный критерий не является простым в применении. Дальнейшее расширение круга задач, к которым можно применять данный критерий, напрямую зависит от успехов в исследовании свойств эндоморфизмов группоидов. В работе формулируются открытые общие проблемы, успехи в исследовании которых расширят возможности применения универсального критерия неподвижной точки. Обсуждается связь между сформулированными проблемами и полученным критерием. Получены необходимые и достаточные условия того, что выполняется гипотеза Римана о нулях дзета-функции Римана. Эти условия получены с помощью универсального критерия неподвижной точки.

Ключевые слова: группоид, эндоморфизм, биполярный тип, биполярная классификация эндоморфизмов группоида, неподвижная точка, универсальный критерий неподвижной точки, гипотеза Римана о нулях дзета-функции, дзета-функция Римана

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Об авторах:

Литаврин Андрей Викторович, канд. физ.-мат. наук, доцент кафедры высшей математики № 2, ИМиФИ ФГАОУ ВО «Сибирский федеральный университет» (660041, Россия, г. Красноярск, пр. Свободный, д. 79), ORCID: https://orcid.org/0000- 0001-6285-0201, anm11@rambler.ru

1. Introduction

Information on *fixed point methods* can be found in the review [1]. Fixed point methods are widely used in functional analysis, differential equations and other areas of mathematics. All fixed point methods are characterized by the identification of sufficient (and sometimes necessary) conditions for some partial transformation f (i.e., a mapping of the form $f : A \to X$, where $A \subseteq X$) of some set X to have a fixed point (i.e., there exists $x \in X$ such that $f(x) = x$). For example, according to Brouwer's fixed point theorem, any continuous mapping of a closed ball into itself (in a finite-dimensional Euclidean space) has a fixed point. Let us denote some well-known theorems that give sufficient conditions for the existence of a fixed point: Hopf's Theorem ([2]), Birkhoff-Kellogg's Theorem ([3]), Schauder's Theorem ([4]). Methods for constructive construction of a fixed point have also been developed. The Banach contraction mapping Theorem (the case of a metric space) is well known.

The main result of the paper is Theorem 3.1 (see Section 2), which gives necessary and sufficient conditions for a point x to be a fixed point of a mapping $f : X \to X$ (i.e. f is a *transformation* of X). Theorem 3.1 is valid for any nonempty set X and any transformation f of X. The necessary and sufficient conditions are formulated using the concept of a *bipolar type of endomorphism of a groupoid* with pairwise distinct left translations. In addition, sufficient conditions are obtained for a point not to be a fixed point of a transformation of some nonempty set. These conditions are expressed as Theorem 3.2 and are formulated using bipolar types of endomorphism of an arbitrary groupoid. The restrictions on left translations disappear. The latter makes Theorem 3.2 easier to use than Theorem 3.1; but the latter gives a stronger statement. The bipolar type of groupoid endomorphism arises in ([5]) in the context of *bipolar classification of groupoid endomorphisms*, which aims to study the following general problems.

Problem 1.1. *For a fixed groupoid G, give an element-wise description of the monoid of all endomorphisms.*

Problem 1.2. *For a fixed groupoid G, give an element-wise description of the group of all automorphisms.*

The fixed point criterion given by Theorem 3.1 is not easy to use. The difficulties in working with it are explained by its universality and problems in working with endomorphisms of a groupoid (as specific transformations). In this paper, we formulate general Problems 4.1–4.5 (see Section 3). Successes in studying these problems should facilitate working with endomorphisms of a groupoid in the context of Theorem 3.1. Consequently, the possibilities in applying the obtained fixed point criterion should be expanded.

The fixed point criterion of Theorem 3.1 is formulated for a transformation. Every partial transformation of any nonempty set can be represented as an ordinary transformation on the new set. In analysis, the connection between fixed points and zeros of some function is well known. All this allows us to apply Theorem 3.1 to the study of the Riemann hypothesis on the zeros of the zeta function. In this paper, we obtain necessary and sufficient conditions for the Riemann hypothesis to be satisfied. The result is expressed as Theorem 5.1 (see Section 4). Theorem 5.1 demonstrates how the results of Theorem 3.1 can be adapted to partial transformations.

Progress in solving Problems 4.1–4.5 may be useful for proving or disproving the Riemann Hypothesis. It is possible that Theorem 5.1 together with accumulated experience with the Riemann zeta function will lead to significant results on its own (without solving the indicated problems). Corollaries 5.1 and 5.2 (see Section 4) of Theorem 5.1 clearly demonstrate that the results of studying Problems 4.1–4.5 can be applied to studying the Riemann Hypothesis.

In 2024, the material [1] «Michael Francis Atiyah. The Riemann Hypothesis» with a proof of the truth of the Riemann hypothesis (open access since 2018) is easily found on the Internet. This work is not published in a peer-reviewed scientific journal (it appeared in the fall of 2018). The author is the outstanding mathematician Michael Francis Atiyah. Criticism of Atiyah's proof is found on the Internet, which is also not published in peer-reviewed scientific journals. A rigorous assessment (and publication of this assessment) of Atiyah's results is complicated by the lack of detail in the proof. Indeed, due to the style of presentation of the material, any assessment of it can only be based on assumptions. The situation is aggravated by Atiyah's death in January 2019. Due to the lack of detail in the proof and the lack of publication in a peer-reviewed journal, the author of this paper considers the Riemann hypothesis about the zeros of the zeta function to be unproven.

2. Basic Definitions and Preliminary Results

A *groupoid* (or *magma*) is a tuple $G = (G, *)$, where $(*)$ is a binary algebraic operation on a set G. The set G is called the *support* of the groupoid $G = (G, *)$. In this paper, the name of the groupoid and its support will be denoted differently where appropriate.

The set of all transformations of a set X will be denoted by $I(X)$ (other notations are also available). The set $I(X)$ is called the *symmetric semigroup of transformations* of the set X (since it is a semigroup with respect to composition). A homomorphism of a groupoid G into itself is called an *endomorphism* of the groupoid G. The set of all endomorphisms of a groupoid G is denoted by $\mathrm{End}(G)$. A transformation α of a groupoid G is an endomorphism of the groupoid G if and only if for any $x, y \in G$ the equality $\alpha(x * y) = \alpha(x) * \alpha(y)$ holds.

In [5], the definition of a bipolar type of endomorphism of a groupoid is introduced. We present this definition and related objects. By $\mathrm{Bte}(G)$ we denote the set of all possible mappings of the set G into the set $\{1, 2\}$. Mappings from this set will be called *bipolar types of endomorphisms* of the groupoid G (or simply *bipolar types*). In [5], for each bipolar type γ, a *base set of endomorphisms* $D(\gamma)$ of the bipolar type γ is introduced (see Definition 3 in [5]). It follows from Theorem 1 of [5] that every endomorphism of an arbitrary groupoid lies in exactly one base set of endomorphisms. Therefore, each endomorphism of a groupoid can be assigned its bipolar type in a unique way (this is the bipolar classification of endomorphisms).

The bipolar type of endomorphism α is denoted by Γ_α. A *left translation* (or an internal translation) of an element $x \in G$ of a groupoid $G = (G, *)$ is a transformation h_x of a set G such that for any $x, y \in G$ the equality $h_x(y) = x * y$ holds. We say that G is a groupoid with *pairwise distinct left translations* if for any $x, y \in G$ the equivalence $h_x = h_y \Leftrightarrow x = y$ holds. Theorem 2.2 from [6] is formulated below as Lemma 2.1.

Lemma 2.1. *Let G be a groupoid with pairwise distinct left translations. Then for an arbitrary $g \in G$ and any endomorphism ϕ of the groupoid G the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) = g, \qquad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \neq g. \tag{2.1}$$

Let G be a groupoid. For each $g \in G$, we introduce the set $M_g := \{m \in G \mid h_m = h_g\}$. Theorem 2.1 from [6] is formulated below as Lemma 2.2.

Lemma 2.2. *Let G be a groupoid. Then for any $g \in G$ and any endomorphism ϕ of the groupoid G, the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) \in M_g, \qquad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \in G \setminus M_g.$$

[1] https://drive.google.com/file/d/17NBICP6OcUSucrXKNWvzLmrQpfUrEKuY/view

3. Universal Fixed Point Criterion

In this section we formulate and prove Theorem 3.1, which will give a universal fixed point criterion for an arbitrary transformation of a set in terms of bipolar types. We will preface it with the preliminary results.

A groupoid $G = (G, *)$ is called a singular semigroup if for any $x, y \in G$ the equality $x * y = x$ holds (more precisely, a left zero semigroup). This groupoid is associative. The following lemma is trivially established.

Lemma 3.1. *On any non-empty set G, we can define a singular semigroup $G = (G, *)$. For a singular semigroup $G = (G, *)$, the equality of sets $\mathrm{End}(G) = I(G)$ holds. The singular semigroup $G = (G, *)$ is a groupoid with pairwise distinct left translations.*

Proposition 3.1. *Let ϕ be some transformation of the set G, $G_1 = (G, *_1)$ and $G_2 = (G, *_2)$ be two groupoids with pairwise distinct left translations such that ϕ is an endomorphism of both groupoids. We assume that $\Gamma_\phi^{G_1}$ is the bipolar endomorphism type ϕ of the groupoid G_1 and $\Gamma_\phi^{G_2}$ is the bipolar endomorphism type ϕ of the groupoid G_2. Then the equality $\Gamma_\phi^{G_1} = \Gamma_\phi^{G_2}$ holds.*

Proof. The assertion follows directly from Lemma 2.1.

In general, the following assertion holds.

Proposition 3.2. *One can specify a set G and its transformation ϕ such that there will exist two groupoids $G_1 = (G, *_1)$ and $G_2 = (G, *_2)$ satisfying the conditions*

$$\phi \in \mathrm{End}(G_1), \quad \phi \in \mathrm{End}(G_2), \quad \Gamma_\phi^{G_1} \neq \Gamma_\phi^{G_2}.$$

Proof. Take the groupoid from Example 1 of [6]. In this example, the base set $D(1,1,2,1)$ contains an endomorphism $(2,2,2,3)$. For this groupoid $\Gamma_\phi(1) = 1$, but 1 is mapped to 2 by $\phi = (2,2,2,3)$. On the other hand, one can construct a singular semigroup with the same support (see Lemma 3.1). In a singular semigroup, the bipolar endomorphism $\phi = (2,2,2,3)$ on 1 will be equal to 2, since 1 is not a fixed point of this endomorphism and the singular semigroup has pairwise distinct left translations (i.e., it satisfies the conditions of Lemma 2.1). The assertion is proved.

Theorem 3.1. *Let G be an arbitrary nonempty set and α be an arbitrary transformation of G. Then a point $g \in G$ is a fixed point of the transformation α if and only if the following conditions are simultaneously satisfied:*
1) there exists a groupoid $G_0 = (G, *)$ with pairwise distinct left translations such that the transformation α is an endomorphism of this groupoid;
2) the equality $\Gamma_\alpha(g) = 1$ holds, where Γ_α is the bipolar type of the endomorphism α of the groupoid G_0.

Proof. Let us show necessity. Let $g \in G$ be an arbitrary fixed point of the transformation α. Introduce a singular semigroup $G_0 = (G, *)$ with support G (this is possible by Lemma 3.1). The transformation α is an endomorphism of the semigroup G_0 (follows from Lemma 3.1). Since all left translations of elements of the semigroup G_0 are distinct, the equality $\Gamma_\alpha(g) = 1$ holds (the latter follows from the equivalence (2.1)). Thus, we have shown that if $g \in G$ is a fixed point of the transformation α, then statements 1 and 2 from the hypothesis of the Theorem hold.

Let us show sufficiency. If statements 1 and 2 from the conditions of the current Theorem are satisfied, then by virtue of the equivalence (2.1) from Lemma 2.1 we obtain that g is a fixed point of the transformation α. The Theorem is proved.

In [5], Ω denotes the bipolar type of a groupoid G such that for any $g \in G$, $\Omega(g) = 2$ (*the second bipolar type*) holds. Theorem 3.1 implies

Corollary 3.1. *A transformation α of a nonempty set G has at least one fixed point if and only if there exists a groupoid $G_0 = (G, *)$ with pairwise distinct left translations such that α is an endomorphism of the given groupoid and the bipolar type of the endomorphism α of this groupoid does not coincide with Ω.*

It follows from Lemma 3.1 that every transformation of the set G is an endomorphism of the singular semigroup with support G. This makes it possible to apply the fixed point criterion from Theorem 3.1 to any transformation of an arbitrary set G. The latter indicates the universality of the above criterion.

If we abandon the condition of pairwise distinctness of left translations in the groupoid, then we can obtain sufficient (but not necessary) conditions for the point $g \in G$ not to be a fixed point of the transformation α.

Theorem 3.2. *Let G be an arbitrary nonempty set and α be an arbitrary transformation of G. We assume that for an element $g \in G$ the following conditions are simultaneously satisfied:*
1) there exists a groupoid $G_0 = (G, *)$ such that the transformation α is an endomorphism of this groupoid;
2) the equality $\Gamma_\alpha(g) = 2$ holds, where Γ_α is the bipolar type of the endomorphism α of the groupoid G_0.

Then the element g is not a fixed point of the transformation α.

Proof. Suppose that g is a fixed point of the transformation α and statements 1 and 2 from the condition of the Theorem hold. Since $\alpha(g) = g$, the set M_g contains the element $\alpha(g)$. Therefore, $\alpha(g) \notin G \setminus M_g$. The latter leads to a contradiction with the assertion of Lemma 2.2. Thus, g is not a fixed point of α. The Theorem is proved.

The above Theorem gives sufficient conditions for a point $g \in G$ not to be a fixed point of α. These conditions are precisely sufficient. They trivially imply necessary conditions for a point g to be a fixed point of α. We formulate these conditions as

Corollary 3.2. *If a transformation α of a nonempty set G has a fixed point $g \in G$, then there is no groupoid with support G and endomorphism α such that $\Gamma_\alpha(g) = 2$ in this groupoid.*

Example 3.1. *Let $\varphi(x) := (1/2)x + 1$ be a transformation of the set $\mathbb{R}$, where $(+)$, $(/)$ are the usual operations on real numbers. Let us show that $x = 2$ is a fixed point of the transformation φ. Consider the groupoid $G = (\mathbb{R}, *)$, where for any $x, y \in \mathbb{R}$ the equality $x * y := x + y - 2$ holds. For any $x, y \in \mathbb{R}$ we have the equalities*

$$\varphi(x * y) = (1/2)(x + y - 2) + 1 = (1/2)(x + y),$$

$$\varphi(x) * \varphi(y) = (1/2)x + 1 + (1/2)y + 1 - 2 = (1/2)(x + y).$$

These equalities show that φ is an endomorphism of the groupoid G. The element $e = 2$ is the neutral element of the groupoid G. It is easy to verify that the groupoid G is a group. Therefore, G has pairwise distinct left translations of elements. It is well known that the neutral element is invariant under all endomorphisms. Therefore e is a fixed point of φ and $\Gamma_\varphi(e) = 1$ (the last equality holds by Lemma 3.1).

Another way to check that $e = 2$ is a fixed point of φ (and the only one at that) is to directly check that $\Gamma_\varphi(e) = 1$. We will do this using equivalence (2) from [6]. The definitions of the sets $L^{(i)}(g)$ can be found in [5] or [6] (type-forming sets). Thus $\Gamma_\varphi(g) = 1$ if and only if $\varphi \in L^{(1)}(g)$. Let $\varphi \in L^{(1)}(x)$ for $x \in \mathbb{R}$. Then for any $y \in \mathbb{R}$ the following equalities hold: $h_{\varphi(x)}(y) = h_x(y)$, $\varphi \cdot h_x = h_x \cdot \varphi$. Each of these equalities holds if and only if $x = 2$. Indeed, the latter follows from the relations:

$$h_{\varphi(x)}(y) = \varphi(x) * y = (1/2)x + 1 + y - 2, \quad h_x(y) = x + y - 2,$$

$$\varphi \cdot h_x(d) = (1/2)(x + d - 2) + 1, \quad h_x \cdot \varphi(d) = ((1/2)d + 1) + x - 2, \quad (\forall d \in \mathbb{R}).$$

Therefore, only for $x = 2$ does the equality $\Gamma_\varphi(x) = 1$ hold, hence $x = 2$ is the only fixed point of the transformation φ (follows from Theorem 3.1).

In Example 3.1 above, we established that 2 is a fixed point of ϕ by using the well-known fact that *a group-to-group homomorphism maps an identity element to an identity element* (in this case, we did not use linear equation solving methods). On the other hand, when we computed $\Gamma_\varphi(g)$ by definition, we were forced to work with linear equations (i.e., we used methods that could by themselves give a fixed point of φ). The latter circumstance illustrates that further investigations of the properties of groupoid endomorphisms and the properties of bipolar types are needed to make productive use of the fixed point criterion given by Theorem 3.1. In the next section we formulate general problems 4.1, 4.2, 4.3, 4.4, and 4.5. Advances in the study of these problems will expand the possibilities of applying the fixed point criterion from Theorem 3.1.

Theorem 3.1 can be applied to a wider range of problems than the statement about the image of the neutral element under a group homomorphism into a group. Indeed, for a particular transformation φ, it is easier to find a groupoid whose endomorphism is ϕ than to construct a group with a similar condition.

Remark 3.1. In recent years, groupoids have found applications. For example, an application of groupoids in biology can be found in [7]. An application of groupoids (more precisely, non-associative groupoids) in cryptography can be found in [8], [9], [10]. There are works aimed at modeling various processes associated with neural networks using groupoids. In this context, the study of the fixed point criterion from Theorem 3.1 seems relevant from the standpoint of algebra applications.

4. Open Problems

For any nonempty set G, let $\mathrm{Gru}(G)$ denote the set of all groupoids with universe G. This set is equivalent to the set $\mathrm{Hom}(G \times G, G)$. For any transformation α of a nonempty set G, let $\mathrm{Gru}(G, \alpha)$ denote the set of all groupoids $S = (G, *)$ in $\mathrm{Gru}(G)$ such that α is an endomorphism of the groupoid S. In the context of Theorem 3.1, there is a natural interest in the following general problem.

Problem 4.1. *For a fixed set G and a fixed transformation α of $I(G)$, solve the following problems.*

(a) Give an element-wise description of the set $\mathrm{Gru}(G, \alpha)$.

(b) Give an element-wise description of all groupoids from $\mathrm{Gru}(G, \alpha)$ *that have pairwise distinct left translations.*

(c) Give an element-wise description of all groupoids $S = (G, *)$ from $\mathrm{Gru}(G, \alpha)$ *such that for any $x \in G$ the transformation inequality $h_x \neq \alpha$ holds, where h_x is the left translation of element x in groupoid S.*

Lemma 3.1 shows that on every non-empty set one can introduce a singular semigroup. The operation in the singular semigroup is structured quite simply, and the set of all endomorphisms coincides with the symmetric semigroup. Consequently, all base sets of endomorphisms of the singular semigroup are non-empty (follows from Lemma 2.1). Therefore, the singular semigroup does not have specificity that can be applied for productive use of Theorem 3.1. There is a natural interest in having a list of all groupoids with pairwise distinct left translations and endomorphism α, which is studied for fixed points. This explains the interest in Problem 4.1, items (a) and (b). Problem 4.1, item (a) is relevant in the context of Theorem 3.2.

If there is a problem in finding fixed points of the transformation α, then this can be explained by the complex conditions of the definition of the transformation α (i.e. it is difficult to find images of elements under the action of this transformation). In order for the difficulties in working with the transformation α not to transfer to working with the groupoid in which the work with the bipolar type of endomorphism α will take place, it is necessary to be able to work in a groupoid, all left translations of which are different from the transformation α itself. This explains the interest in Problem 4.1, point (c).

There are situations when the transformation α is specified by simple conditions (i.e. there is no fundamental difficulty in finding the α-images of elements). But the method of specifying the transformation α is not convenient for solving the corresponding equations. For example, the Riemann zeta function can be attributed to such partial transformations (partial transformations can be reduced to ordinary transformations of a new set). The Riemann zeta function is defined by fairly simple conditions and there is no fundamental difficulty in calculating its values on specific complex numbers. At the same time, the Riemann zeta function zeros hypothesis is associated with it (the function zeros and fixed points are closely related, see the next section). The latter indicates that the Riemann zeta function is simply inconvenient for solving equations written with its help.

Problem 4.1 is a general problem. It can be seen as a template for similar problems solved for a particular set G and a particular transformation α. In this case, together with Theorem 3.1, the results of solving Problem 4.1 will be useful for finding fixed points of a particular transformation α. Problem 4.1 can also be investigated in a general form. The results of such investigations will be useful for solving Problem 4.1 for particular G and α. Partial solutions of Problem 4.1 (i.e., not all groupoids with the required properties are found) can also be useful in the context of Theorem 3.1. An element-wise description of a groupoid in Problem 4.1 is any description of a groupoid that allows one to write out left translations of this groupoid (i.e., to define its arithmetic).

For a particular groupoid $G = (G, *)$, let $\mathrm{Bte}^*(G)$ denote the subset of the set $\mathrm{Bte}(G)$ defined by the equivalence: $\gamma \in \mathrm{Bte}^*(G) \Leftrightarrow D(\gamma) \neq \varnothing$. For any groupoid G, the set $\mathrm{Bte}^*(G)$

is not empty. Indeed, it always contains the first bipolar type A (i.e., $A(g) = 1$ for any $g \in G$).

Problem 4.2. *For a fixed groupoid G, give a description of the elements of set* $\mathrm{Bte}^*(G)$.

Problem 4.2 is relevant in the context of Problem 1.1 (describing the monoid of all endomorphisms) and Problem 1.2 (describing the group of all automorphisms) without the condition of pairwise distinctness of left translations. Solving Problem 4.2 may cause difficulties even for groupoids for which Problem 1.1 has been solved. In the context of Theorem 3.1, the following proposition indicates the relevance of solving Problem 4.2.

Proposition 4.1. *Let $G = (G, *)$ be a groupoid with pairwise distinct left translations, $g \in G$ be some element of G, and α be an endomorphism of G distinct from the identity transformation. If the set $\mathrm{Bte}^*(G) \setminus \{A\}$ does not contain a bipolar type γ such that $\gamma(g) = 1$, then g is not a fixed point of the endomorphism α.*

Proof. Since G is a groupoid with pairwise distinct left translations, the base set of $D(A)$ (of endomorphisms of the first bipolar type) consists only of the identity transformation (this follows directly from Lemma 2.1). Therefore, the endomorphism α from the conditions of the proposition is not contained in $D(A)$. By the conditions of the proposition, the endomorphism α is contained in the base set of endomorphisms $D(\gamma)$ such that $\gamma(g) \neq 1$ (hence, $\gamma(g) = 2$). Therefore, g is not a fixed point of the endomorphism α. Indeed, the latter follows from both Theorem 3.1 and Theorem 3.2 (independently). The proposition is proved.

In this case, it is not important for us to know the bipolar type of the endomorphism α. It is enough to know that α is an endomorphism of the groupoid G that is distinct from the identity transformation. Problem 4.2 can be considered for specific groupoids G. It is also relevant to study Problem 4.2 in the general case.

Let $G = (G, *)$ be an arbitrary groupoid and A be the first bipolar type. Let us distinguish subsets $\mathrm{APriori}(G, 1)$ and $\mathrm{APriori}(G, 2)$ in the set G:

$$\mathrm{APriori}(G, i) := \{g \in G \mid \forall \gamma \in \mathrm{Bte}^*(G) \setminus \{A\} : \ \gamma(g) = i\}, \quad i = 1, 2.$$

The exclusion of the bipolar type A from the set $\mathrm{Bte}^*(G)$ is done so that the set $\mathrm{APriori}(G, 2)$ can be calculated. The base set of endomorphisms of the first type is always non-empty. Therefore, if we do not remove the first bipolar type from the conditions introducing $\mathrm{APriori}(G, 2)$, then $\mathrm{APriori}(G, 2)$ will always be the empty set. The values of all bipolar types from $\mathrm{Bte}^*(G) \setminus \{A\}$ on elements of $\mathrm{APriori}(G, 2)$ and $\mathrm{APriori}(G, 2)$ are known a priori. Indeed, this follows directly from the definition of these sets.

Problem 4.3. *For a fixed groupoid G, give an element-wise description of the sets* $\mathrm{APriori}(G, 1)$ *and* $\mathrm{APriori}(G, 2)$.

The solution of the above problem can be difficult even if we know the element-wise description of the monoid of all endomorphisms of the groupoid G.

Problem 4.4. *For a fixed set G and a fixed transformation α of set G, solve the following problems:*

(a) For a fixed set $H \subseteq G$, give a description of all groupoids $S = (G, *)$ from $\mathrm{Gru}(G, \alpha)$ *such that $H \subseteq \mathrm{APriori}(S, 1)$.*

(b) For a fixed set $H \subseteq G$, give a description of all groupoids $S = (G, *)$ from $\mathrm{Gru}(G, \alpha)$ *such that $H \subseteq \mathrm{APriori}(S, 2)$.*

In Example 3.1 we used the fact that the neutral element of a group is a fixed point for any endomorphism. Hence, $e \in \mathrm{APriori}(G, 1)$ when G is a group. Thus, successes in solving Problems 4.3 and 4.4 will allow us to construct (or select, if there are successes in solving Problem 4.1) groupoids from the conditions of Theorem 3.1 (analogously to Theorem 3.2) such that the analysis of the condition $\Gamma_\alpha(g) = 1$ (analogously to $\Gamma_\alpha(g) = 2$) will be significantly simplified. Successes in investigating Problem 4.4, item (a), are not necessarily necessary to lead to success in investigating item (b), and vice versa.

In [11], the possibility of computing the bipolar type of the composition $\phi \cdot \psi$ of two endomorphisms ϕ and ψ of some groupoid G is investigated using the bipolar types Γ_ϕ and Γ_ψ (i.e., it is necessary to compute $\Gamma_{\phi\cdot\psi}$ without computing the composition $\phi\cdot\psi$). In [11], the concept of *alternating pair of endomorphisms* arises. Thus, a pair of endomorphisms (ψ, ϕ) of a groupoid G is called alternating if for any $g \in G$ the condition $(\Gamma_\phi(g), \Gamma_\psi(\phi(g))) \neq (2, 2)$ is satisfied. For every alternating pair of endomorphisms, the equality $\Gamma_{\phi\cdot\psi}(g) = \Gamma_\phi(g) \times \Gamma_\psi(\phi(g))$ holds, where $(\times)$ is the product of two natural numbers (see Theorem 2 in [11]). For non-alternating pairs of endomorphisms, the above equality obviously does not hold (since the values of the bipolar type on an element are 1 or 2). In the context of this result, the following general problem is of interest.

Problem 4.5. *Let $G = (G, *)$ be an arbitrary groupoid and (ψ, ϕ) be an arbitrary non-alternating pair of endomorphisms of the groupoid G. For all $g \in G$ such that the equality $(\Gamma_\phi(g), \Gamma_\psi(\phi(g))) = (2, 2)$ holds, compute $\Gamma_{\phi\cdot\psi}(g)$.*

Example 1 of [11] suggests that the solution to Problem 4.5 will be considerably more difficult than the analogous result for alternating pairs of endomorphisms. The solution to Problem 4.5 provides additional properties that allow one to calculate the value of the bipolar type of an arbitrary endomorphism on elements of a groupoid. Therefore, progress in the study of Problem 4.5 may be useful in the context of Theorems 3.1 and 3.2.

Remark 4.1. Each of Problems 4.1–4.5 can be solved for specific parameters (for example, a specific G and a specific α) that define the problem (let's call such solutions – particular solutions). These problems can also be solved in the general case. In the latter case, it is necessary to use systems of mathematical objects that allow working with arbitrary groupoids. Problems 4.1–4.5 in the general case can be solved differently (i.e., the solution will be implemented using different systems of mathematical objects). In the context of Theorem 3.1, different solutions to one problem can be relevant. Thus, one solution is conveniently applied to one class of specific problems, and another solution to another class. The latter also applies to the general Problems 1.1 and 1.2.

Remark 4.2. Corollaries 5.1 and 5.2 from the next section indicate a direct connection between Problems 4.1–4.4 and the Riemann hypothesis on the zeros of the zeta function.

5. Riemann's hypothesis on the zeros of the zeta function

In this section, Theorem 5.1 will be formulated and proved. We will preface this Theorem with the necessary theoretical information about the zeta function and Lemma 5.1.

The Riemann zeta function $\zeta(s)$ is defined for any $s \in \mathbb{C}$ except for the number $s = 1$. Therefore, the zeta function $\zeta(s)$ is a partial transformation of the set $\mathbb{C}$. If $\{x\} := x - [x]$ is the fractional part of a number $x \in \mathbb{R}$, then the values of the zeta function can be obtained from the equalities (see paper [12]):

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1;$$

$$\zeta(s) = \frac{s}{s-1} + s\int_1^{+\infty} \frac{\{u\}}{u^{s+1}} du, \quad 0 < \mathrm{Re}(s) \leq 1,\ s \neq 1. \tag{5.1}$$

Information on the values of the zeta function on other complex numbers are not required in this paper. This information can be found, for example, in the papers: [12],[13],[14] and [15]. Other methods for calculating the zeta function values can also be found there.

Non-trivial zeros of the zeta function ζ are the zeros of ζ that lie in the strip $0 \leq \mathrm{Re}(s) \leq 1$ (this set of complex numbers is called the *critical strip*). The line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \frac{1}{2}\}$ is called the *critical line*.

Riemann Hypothesis. All non-trivial zeros of the Riemann zeta function lie on the critical line.

If z is a zero of the zeta function, then the zeros of the zeta function are complex numbers: $1 - z, \overline{z}, 1 - \overline{z}$ (see [12]). The zeros of the zeta function do not lie on the lines: $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. The absence of zeros of the zeta function on the line $\mathrm{Re}(s) = 1$ follows from Theorem 7.1 in the review paper [15]. If the line $\mathrm{Re}(s) = 0$ contains a zero $z = it$ of the zeta function, then $1 - z = 1 - (0 + it) = 1 - it$ is a zero of the zeta function. The latter is impossible, in view of what was said above. Thus, to study the Riemann hypothesis, it will be sufficient for us to define the zeta function ζ on the set $0 < \mathrm{Re}(s) < 1$ (for example, using the relation (5.1)).

Auxiliary set and transformation. Next, we introduce the set $\mathbb{C}^o$ and the transformation Θ on it, which will help us apply Theorem 3.1 to study the properties of the zeros of the zeta function.

By $\diamond$ we denote the symbol, which by definition is not a complex number. Next, we introduce the set $\mathbb{C}^o := \mathbb{C} \cup \{\diamond\}$. As usual, $(\setminus)$ is the difference of sets. On the set $\mathbb{C}^o$, we define the transformation

$$\Theta(s) := \begin{cases} \zeta(s) + s & s \in \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\} \\ \diamond, & s \in \mathbb{C}^o \setminus \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}. \end{cases} \tag{5.2}$$

The mapping $\Theta(s)$ is a transformation of the set $\mathbb{C}^o$. It follows from (5.2) that $\diamond$ is a fixed point of the transformation $\Theta(s)$. For any complex number s from the set $\{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$ the following implications hold:

$$\zeta(s) = 0 \ \Rightarrow\ \Theta(s) = \zeta(s) + s = s, \quad \Theta(s) = s \ \Rightarrow\ \zeta(s) + s = s \ \Rightarrow\ \zeta(s) = 0.$$

Therefore, the point $s \in \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$ is a fixed point of Θ if and only if it is a nontrivial zero of the Riemann zeta function $\zeta(s)$. The connection between a fixed point of a function and a zero of the function used above is well known in analysis. Since $\diamond$ by definition does not belong to the set $\mathbb{C}$, then for any point $s \in \mathbb{C} \setminus \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$ the condition $\Theta(s) \neq s$ holds. Thus, the following lemma is proved.

Lemma 5.1. *Every fixed point of Θ is either $\diamond$ or a nontrivial zero of the Riemann zeta function. Every nontrivial zero of the Riemann zeta function is a fixed point of Θ.*

Theorem 5.1. *The Riemann Hypothesis is true if and only if the following statements hold.*
1) There exists a groupoid $(\mathbb{C}^o, *)$ with pairwise distinct left translations such that the transformation Θ is an endomorphism of this groupoid.
2) For any point $s \in \mathbb{C}^o$ the following implications hold:

$$\Gamma_\Theta(s) = 1 \Rightarrow \left(\mathrm{Re}(s) = \frac{1}{2}\right) \vee (s = \diamond), \quad \left(\mathrm{Re}(s) \neq \frac{1}{2}\right) \wedge (s \neq \diamond) \Rightarrow \Gamma_\Theta(s) = 2, \tag{5.3}$$

where Γ_Θ is the bipolar endomorphism type Θ of the groupoid $(\mathbb{C}^o, *)$.

Proof. Suppose the Riemann hypothesis. Then for every nontrivial zero s of the zeta function ζ the equality $\mathrm{Re}(s) = \frac{1}{2}$ holds. By Lemma 5.1 the transformation Θ has fixed points (at least one point: $\diamond$). Therefore, by Theorem 3.1, there exists a groupoid $\mathbb{C}^o = (\mathbb{C}^o, *)$ with pairwise distinct left translations and an endomorphism Θ. Moreover, the equality $\Gamma_\Theta(s) = 1$ holds if and only if $s \in \mathbb{C}^o$ is a fixed point of the endomorphism Θ of the groupoid $\mathbb{C}^o$ (the latter follows from Theorem 3.1). Since the fixed points of the transformation Θ are exhausted by the nontrivial zeros of the zeta function and the symbol $\diamond$, then if $s \in \mathbb{C}^o$ is a fixed point of Θ, then either $s = \diamond$ or $\mathrm{Re}(s) = \frac{1}{2}$. Therefore, the first implication (5.3) holds. This implication cannot be replaced by an equivalence, since the critical line contains not only the zeros of the zeta function. The second implication of (5.3) follows from the assumption that the Riemann hypothesis is true.

Let statements 1 and 2 from the statement of the current Theorem be true. Consequently, any fixed point of the transformation Θ lies on the critical line or is a symbol of $\diamond$ (deduced from Theorem 3.1). Therefore, Lemma 5.1 implies that all nontrivial zeros of the zeta function lie on the critical line (i.e., the Riemann hypothesis is true). The Theorem is proved.

The following corollaries follow from Theorem 5.1 and the definitions of the sets $\mathrm{Gru}(G, \alpha)$, $\mathrm{APriori}(S, 1)$, and $\mathrm{Bte}^*(S)$.

Corollary 5.1. *If there exists a groupoid $S \in \mathrm{Gru}(\mathbb{C}^o, \Theta)$ with pairwise distinct left translations and there exists a set*

$$H \subseteq \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1, \mathrm{Re}(s) \neq \frac{1}{2}\},$$

such that $H \subseteq \mathrm{APriori}(S, 1)$, then the Riemann hypothesis does not hold.

If the premises of Corollary 5.1 hold, then there exist nontrivial zeros of the zeta function that do not lie on the critical line. In this case, the Riemann hypothesis does not hold.

Corollary 5.2. *If there exists a groupoid $S \in \mathrm{Gru}(\mathbb{C}^o, \Theta)$ with pairwise distinct left translations such that for any $\gamma \in \mathrm{Bte}(S)$ the implication*

$$\gamma \in \mathrm{Bte}^*(S) \setminus \{A\} \Rightarrow \left(\forall s \in \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\} : \gamma(s) = 1 \Rightarrow \mathrm{Re}(s) = \frac{1}{2}\right),$$

then the Riemann hypothesis holds.

If the premises of Corollary 5.2 hold, then there exists a groupoid $S \in \mathrm{Gru}(\mathbb{C}^o, \Theta)$ with pairwise distinct left translations such that all endomorphisms, except those lying in $D(A)$, of this groupoid have fixed points only on the critical line. Since S is a groupoid with pairwise distinct left translations, the base set of endomorphisms $D(A)$ contains only the identity transformation. In this case, the Riemann Hypothesis holds.

Corollaries 5.1 and 5.2 demonstrate that the solution of Problems 3-6 for $G = \mathbb{C}^o$ and $\alpha = \Theta$ is useful for investigating the Riemann Hypothesis.

If the Riemann hypothesis is not satisfied, then the existence of the groupoid S from Corollary 5.1 does not follow. Similarly, the existence of the groupoid S from Corollary 5.1 does not follow from the Riemann hypothesis. At the same time, the author of this paper is inclined to believe that the following hypotheses are satisfied.

Hypothesis 5.1. *If the Riemann hypothesis is not satisfied, then the groupoid S from* $\mathrm{Gru}(\mathbb{C}^o, \Theta)$ *from the conditions of Corollary* 5.1 *exists.*

Hypothesis 5.2. *If the Riemann hypothesis is satisfied, then the groupoid S from* $\mathrm{Gru}(\mathbb{C}^o, \Theta)$ *from the conditions of Corollary* 5.2 *exists.*

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Автор прочитал и одобрил окончательный вариант рукописи.
Конфликт интересов: автор заявляет об отсутствии конфликта интересов.